Journal of Algebra **235,** 275–314 (2001)
doi:10.1006/jabr.2000.8485, available online at http://www.idealibrary.com on IDEAL®

The Schur Indices of the Irreducible Characters of the Special Linear Groups

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Communicated by Walter Feit

Received March 10, 2000

We compute the Schur indices of each irreducible character of $\mathbf{SL}(n, q)$ the special linear group, for all $n \geq 1$ and for all q a power of a prime. © 2001 Academic Press

Key Words: Brauer group; Schur index; finite general linear groups; finite special linear groups; characters; representations.

1. INTRODUCTION

The Schur index was introduced by Schur in 1905. If χ is an irreducible character of a finite group G and F is a field of characteristic zero, the Schur index $\mathrm{m}_F(\chi)$ of χ with respect to F is the smallest positive multiplicity of χ in the character afforded by a G-module over the field F. Schur calculated the Schur indices for the characters of a number of classical groups, and ever since their calculation has been sought for the characters of the classical groups. The Schur indices are intimately connected to elements of certain Brauer groups. Brauer's important work on Schur indices elucidated this connection and other fundamental properties of Schur indices. Further results on Schur indices have been obtained by many others including Feit [3, 4]. An overview of some of the known general properties of the Schur index can be found in [3].

In this paper, we calculate the Schur indices of all the irreducible characters of $\mathbf{SL}(n, q)$, the finite special linear groups, for all $n \geq 1$ and q a power of a prime. Our proof is inspired by results in [16] and relies on the formula to calculate certain Schur indices obtained in [17].

[1] Partially supported by a grant from the NSA.

0021-8693/01 $35.00

The Schur indices have been completely calculated for some families of finite quasisimple classical groups. Work by Feit [3, 4] calculated the Schur indices of the characters of all the simple groups of order less than 10^6 and of all the covering groups of all the sporadic simple groups. I calculated in [15] those of the covering groups of the symmetric and alternating groups. Those of other families of classical groups are calculated in [1, 18]. In 1955, Green [7] calculated the irreducible characters of $\mathbf{GL}(n, q)$, the general linear group of degree n over a finite field. The Schur indices of the general linear group were finally computed by Zelevinsky [18] in 1981. Lehrer [10] described the parameterization of the characters of $\mathbf{SL}(n, q)$ in 1973. Partial results on the calculation of the Schur indices for the irreducible characters of the special linear groups have been obtained by Janusz [8], Ohmori [12], Gow [5, 6], and Prasad [14]. The most detailed information on the Schur indices of the irreducible characters of $\mathbf{SL}(n, q)$ prior to the present paper appears in [6]. The paper [14] adds some information on the real Schur indices of the irreducible characters of $\mathbf{SL}(6,q)$.

We use Lehrer's parameters for each irreducible character of $\mathbf{SL}(n, q)$ to give an algorithm to calculate each Schur index. To make explicit this parameterization, we define some combinatorial objects $\mathscr{F}_n$, and we have a bijection between $\mathscr{F}_n$ and $\mathbf{Irr}(\mathbf{GL}(n, q))$, the set of irreducible characters of $\mathbf{GL}(n, q)$; see Theorem 2.4 below. Associated with each $\lambda \in \mathscr{F}_n$, we have an irreducible character $\chi_\lambda \in \mathbf{Irr}(\mathbf{GL}(n, q))$ and an irreducible character $\psi_\lambda \in \mathbf{Irr}(\mathbf{SL}(n, q))$. The ψ_λ run through all the irreducible characters of $\mathbf{SL}(n, q)$ up to $\mathbf{GL}(n, q)$ conjugacy. We have two natural actions of $\mathscr{F}_n$, an action corresponding to the Galois action on χ_λ, and an action corresponding to the multiplication of χ_λ by linear characters of $\mathbf{GL}(n, q)$.

We give two answers to the problem of calculating the Schur indices. The first answer, Theorem 5.9 below, calculates for each $\lambda \in \mathscr{F}_n$ the element $[\psi_\lambda]$ of the Brauer group in terms of a cross product of objects defined combinatorially from λ and its two acting groups. The Schur index $\mathrm{m}_F(\psi_\lambda) = 1$ if and only if $[\psi_\lambda] = 1$. When $[\psi_\lambda] \neq 1$, then $\mathrm{m}_F(\psi_\lambda) = 2$.

Our second answer relies on local Schur indices. For p a rational prime or ∞, the local Schur index $\mathrm{m}_p(\psi_\lambda) = \mathrm{m}_{\mathbf{Q}_p}(\psi_\lambda)$ is simply the Schur index of ψ_λ over the field $\mathbf{Q}_p$ of p-adic numbers (the field $\mathbf{R}$ of real numbers if $p = \infty$); see [3] for details. We define

$$M(\psi_\lambda) = \left\{p : p \text{ is a rational prime or } \infty \text{ and } \mathrm{m}_p(\psi_\lambda) \neq 1\right\}.$$

Since the Schur indices involved are all 1 or 2 in our case, the set $M(\psi_\lambda)$ characterizes the element $[\psi_\lambda] \in \mathrm{Br}(\mathbf{Q}(\psi_\lambda))$. In our second answer, we give an explicit algorithm to calculate, from λ, the set $M(\psi_\lambda)$. This algorithm first yields a finite set M such that $M(\psi_\lambda) \subseteq M$ and then allows for the explicit calculation of $m_p(\psi_\lambda)$ for each prime p and, in particular, for each prime p in the finite set M. The set M is described in Theorem 9.3. The individual calculations of local Schur indices are given in Theorems 6.5, 7.3, and 8.8.

This paper settles a number of questions about the Schur indices of the special linear groups. If $\psi \in \mathbf{Irr}(G)$ and G is one of the finite classical groups studied in [3, 4], then $|M(\psi)| \leq 2$. The groups studied in [3, 4] include all the simple groups of order less than 10^6, the groups $\mathbf{SL}(2, q)$, and all the sporadic simple groups and their covering groups. On the other hand, $|M(\psi)|$ is unbounded when G runs through the covering groups of the symmetric and alternating groups (see [15]) or through the covering groups of the Weyl groups of type B_n; see [1]. The work of Gow [6], which provided the most detailed information on the Schur indices for $\mathbf{SL}(n, q)$ prior to the present paper, left open the question as to whether or not there existed a $\psi \in \mathbf{Irr}(\mathbf{SL}(n, q))$ such that $|M(\psi)| > 2$, and even whether or not there could be some $\psi \in \mathbf{Irr}(\mathbf{PSL}(n, q))$ with $M(\psi) \neq \emptyset$. As a corollary of our results, we obtain here that $|M(\psi)|$ is in fact unbounded even for $\psi \in \mathbf{Irr}(\mathbf{PSL}(n, q))$; see Corollary 9.5.

I thank Kevin Keating for a number of useful conversations on number theoretical topics relevant to the present paper.

2. GENERAL NOTATION

Quite often in this paper, we need to refer to the p-part of an integer n, where p is a prime. By this we just mean the largest power of p dividing n, and we denote it by n_p. In particular, the 2-part n_2 of n plays a prominent role in many of the statements of our results.

The irreducible characters of $\mathbf{GL}(n, q)$ have been described by Green [7]. We will use notation suggested by Lehrer [10] to parameterize the characters of $\mathbf{GL}(n, q)$.

We let p be a prime number and we let q be a power of p. We have the finite field $\mathbf{F}_p$, with exactly p elements. We fix an algebraic closure of $\mathbf{F}_p$, which we denote $\overline{\mathbf{F}_p}$. We think of each finite field in characteristic p as a subfield of $\overline{\mathbf{F}_p}$. In particular, $\mathbf{F}_q$ is a subfield of $\overline{\mathbf{F}_p}$. For each positive integer d, we denote by $F_d = \mathbf{F}_{q^d}^\times$ the multiplicative group of the field $\mathbf{F}_{q^d}$. We denote by $\widehat{F_d}$ the character group of F_d, that is, $\widehat{F_d}$ is the group of group homomorphisms $F_d \to \mathbf{C}^\times$. We define $\sigma_q \colon \widehat{F_d} \to \widehat{F_d}$ by $\sigma_q(\theta) = \theta^q$.

DEFINITION 2.1. (1) Two characters θ and ϕ in $\widehat{F_d}$ are *conjugate* if $\sigma_q{}^k(\theta) = \phi$ for some integer k. This yields an equivalence relation.

(2) A *d-simplex* s is a conjugacy class of size d in $\widehat{F_d}$. If θ is some element of s, we write $s = \langle \theta \rangle$.

(3) The *degree* of a d-simplex s is $d(s) = d$.

(4) We denote by $\mathscr{G}_d$ the union of all the d-simplices and by $\mathscr{G}$ the union $\bigcup_{d=1}^{\infty} \mathscr{G}_d$.

We let n be a positive integer. Then $\mathbf{F}_{q^n}$ can be viewed as a vector space over $\mathbf{F}_q$ of dimension n. We set $\mathbf{GL}(n, q)$ to be the group of all invertible linear transformations of the vector space $\mathbf{F}_{q^n}$ over $\mathbf{F}_q$ onto itself. Each element of F_n acts on the vector space $\mathbf{F}_{q^n}$ by left multiplication, so we think of F_n as a subgroup of $\mathbf{GL}(n, q)$. There is the determinant function **det** which provides a surjective homomorphism

$$\mathbf{det}\colon \mathbf{GL}(n, q) \to F_1,$$

whose kernel is denoted $\mathbf{SL}(n, q)$, and which provides a fixed isomorphism from $\mathbf{GL}(n, q)/\mathbf{SL}(n, q)$ onto F_1.

Let $\mathscr{P}$ be the set of all partitions. For each $\nu \in \mathscr{P}$, we denote, as usual, by $|\nu|$ the sum of its parts. Then we have the following fundamental theorem, essentially due to Green [7]. We use the details of the parameterization as given in Theorem 5 of [10].

THEOREM 2.2. *Let $\mathscr{S}$ be the set of d-simplexes $(d = 1, 2, \dots)$. A partition-valued function $\lambda\colon \mathscr{S} \to \mathscr{P}$ such that $\sum_{\langle \theta \rangle \in \mathscr{S}} d(\langle \theta \rangle) \,|\, \lambda(\langle \theta \rangle)| = n$ determines a unique irreducible character of $\mathbf{GL}(n, q)$ denoted $(\cdots \langle \theta \rangle^{\lambda\langle \theta \rangle} \cdots)$, and all irreducible characters arise uniquely in this way.*

Proof. See Theorem 5 in [10] for a detailed description of the parameterization and a proof of this theorem.

Since we will need to refer often to this parameterization, we introduce some convenient notation to describe it.

DEFINITION 2.3. (1) We let $\mathscr{F}$ be the set of all functions $\lambda\colon \mathscr{G} \to \mathscr{P}$, which assign the empty partition to almost all elements of $\mathscr{G}$ and have the property that, for every $\theta \in \mathscr{G}$, we have $\lambda(\sigma_q(\theta)) = \lambda(\theta)$.

(2) For each $\lambda \in \mathscr{F}$, we define its *degree* to be

$$\mathbf{deg}(\lambda) = \sum_{\theta \in \mathscr{G}} |\lambda(\theta)|.$$

(3) We denote by $\mathscr{F}_n$ the set of all elements of $\mathscr{F}$ of degree n.

(4) For each $\lambda \in \mathscr{F}_n$, we denote by χ_λ the irreducible character $(\cdots \langle \theta \rangle^{\lambda(\theta)} \cdots)$, where we take $\langle \theta \rangle^{\lambda(\theta)}$ once for each simplex. Hence, $\chi_\lambda \in \mathbf{Irr}(\mathbf{GL}(n, q))$.

THEOREM 2.4. *The map $\lambda \mapsto \chi_\lambda$ is a bijection $\mathscr{F}_n \to \mathbf{Irr}(\mathbf{GL}(n, q))$.*

Proof. This is just a restatement of Theorem 2.2.

If $\lambda \in \mathscr{F}_n$, then the restriction of $\chi_\lambda \in \mathbf{Irr}(\mathbf{GL}(n, q))$ to the center $\mathbf{Z}(\mathbf{GL}(n, q))$ of $\mathbf{GL}(n, q)$ is a multiple of some irreducible linear character.

This linear character plays an important role in what follows. We include, for completeness, an algorithm to compute it from λ. The map $\alpha \mapsto \alpha \cdot 1$ provides a natural isomorphism from F_1 onto $Z(\mathbf{GL}(n,q))$. Hence, the linear character in question can be thought of as simply being an element of $\widehat{F_1}$.

PROPOSITION 2.5. *Let $\lambda \in \mathscr{F}_n$. Then the restriction of χ_λ to $Z(\mathbf{GL}(n,q))$ is a multiple of some linear character ρ. Furthermore, this linear character ρ, when viewed as an element of $\widehat{F_1}$, can be computed as*

$$\rho = \prod_{d=1}^{\infty} \prod_{\theta} \left(\mathrm{Res}_{F_1}^{F_d}(\theta)\right)^{|\lambda(\theta)|},$$

where, in the second product, θ runs through a set of representatives of the d-simplices.

Proof. Note that $\sigma_q(\theta)$ and θ have the same restriction to F_1, so the product is independent of the chosen set of representatives. The result follows easily from the description of the parameterization given in [10].

For each positive integer m we denote by $\eta_m = \mathbf{exp}(2\pi i/m)$ a fixed primitive mth root of 1. For each integer r which is relatively prime to m we set $\sigma_r \in \mathrm{Gal}(\mathbf{Q}(\eta_m)/\mathbf{Q})$ to be the unique automorphism of $\mathbf{Q}(\eta_m)$ such that $\sigma_r(\eta_m) = \eta_m^r$. Note that this notation meshes with the one used in Definition 2.1. With the notation there, and setting $m = |\theta(F_d)|$, we have, since $(q,m) = 1$, that $\sigma_q \in \mathrm{Gal}(\mathbf{Q}(\eta_m)/\mathbf{Q})$, and the composition of the character θ with the Galois automorphism σ_q is simply $\sigma_q\theta = \sigma_q(\theta)$ in the notation before Definition 2.1.

DEFINITION 2.6. (1) For each $\lambda \in \mathscr{F}$, we denote by $\mathbf{Q}(\lambda)$ the field $\mathbf{Q}$ extended by the values of all the θ in the support of λ. Hence, $\mathbf{Q}(\lambda)$ is $\mathbf{Q}$ extended by a primitive mth root of 1, where m is the least common multiple of all the $|\theta(F_d)|$ for $\theta \in \mathscr{G}_d$ such that $\lambda(\theta)$ is not the empty partition.

(2) Let $\lambda \in \mathscr{F}_n$ and let $\sigma \in \mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$. Then, we define $\sigma\lambda$: $\mathscr{G} \to \mathscr{P}$ by, for $\theta \in \mathscr{G}$, setting $\sigma\lambda(\theta) = \lambda(\sigma^{-1}\theta)$ if θ is such that $\mathbf{Q}(\theta) \subseteq \mathbf{Q}(\lambda)$, and by setting $\sigma\lambda(\theta)$ to be the empty partition otherwise. Naturally, here $\sigma^{-1}\theta$ denotes function composition. We have that $\sigma\lambda \in \mathscr{F}_n$.

(3) Let $\lambda \in \mathscr{F}_n$. We set

$$\mathrm{Galg}(\lambda) = \{\sigma \in \mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q}) : \sigma\lambda = \lambda\}.$$

(4) For any subgroup H of $\mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$, we denote by H' the fixed field of H.

Remark. Galg(λ) is a subgroup of Gal($\mathbf{Q}(\lambda)/\mathbf{Q}$) which can easily be computed from λ. The g in Galg stands for the g in general linear group.

LEMMA 2.7. *Let* $\lambda \in \mathscr{F}_n$. *Then* $\mathbf{Q}(\chi_\lambda) \subseteq \mathbf{Q}(\lambda)$ *and, for each* $\sigma \in$ Gal($\mathbf{Q}(\lambda)/\mathbf{Q}$), *we have that the composition of the character with the Galois automorphism is simply* $\sigma\chi_\lambda = \chi_{\sigma\lambda}$; *see Definition* 2.6.

Proof. This follows from the description of the character χ_λ given in [10].

PROPOSITION 2.8. *Let* $\lambda \in \mathscr{F}$. *Then the field of values of* χ_λ *is*

$$\mathbf{Q}(\chi_\lambda) = \mathbf{Q}(\lambda)^{\mathrm{Galg}(\lambda)} = \mathrm{Galg}(\lambda)',$$

in other words, it is simply the fixed field of Galg(λ).

Proof. Since $\mathbf{Q}(\lambda)/\mathbf{Q}$ is a finite Galois extension, this follows immediately from Lemma 2.7 and Theorem 2.4.

3. MULTIPLYING IRREDUCIBLE CHARACTERS OF $\mathbf{GL}(n, q)$ BY LINEAR CHARACTERS

As remarked above, the determinant function **det** provides a fixed isomorphism from $\mathbf{GL}(n, q)/\mathbf{SL}(n, q)$ onto F_1. If $\alpha \in \widehat{F_1}$, then $\alpha\mathbf{det}$ is a linear character of $\mathbf{GL}(n, q)$. Building on some results of Lehrer [10], Karkar and Green [9] have described the multiplication action of this character on $\mathbf{Irr}(\mathbf{GL}(n, q))$. We introduce notation to describe their result, and to study certain aspects of the multiplication action and its interaction with the Galois action introduced earlier.

DEFINITION 3.1. Let $\alpha \in \widehat{F_1}$. Then, we wish to define the action of α on various objects.

(1) By abuse of notation, we may also view α as a linear character of $\mathbf{GL}(n, q)$ (strictly speaking, as the composition $\alpha\mathbf{det}$) or as a linear character of F_d for any positive integer d (strictly speaking, as the composition αNorm, where Norm is the norm homomorphism Norm: $\mathbf{F}_{q^d}^\times \to \mathbf{F}_q^\times$). The context will determine which version of α needs to be used.

(2) If $\chi \in \mathbf{Irr}(\mathbf{GL}(n, q))$, then $\alpha\chi$ is simply the product of the two characters of $\mathbf{GL}(n, q)$.

(3) If $\theta \in \widehat{F_d}$, for some positive integer d, then $\alpha\theta$ is simply the product of the two elements of $\widehat{F_d}$.

(4) If $\lambda \in \mathscr{F}$, then we define $\alpha\lambda$: $\mathscr{G} \to \mathscr{P}$ by $\alpha\lambda(\theta) = \lambda(\alpha^{-1}\theta)$. It is easy to see that $\alpha\lambda \in \mathscr{F}$ and $\mathbf{deg}(\alpha\lambda) = \mathbf{deg}(\lambda)$.

(5) If $\lambda \in \mathscr{F}$, then we define the subgroup of F_1

$$\mathscr{I}(\lambda) = \bigcap \left\{ \ker(\alpha) : \alpha \in \widehat{F_1} \text{ and } \alpha\lambda = \lambda \right\}.$$

(6) Let $\lambda \in \mathscr{F}_n$. We set

$$\mathrm{Galr}(\lambda) = \left\{ \sigma \in \mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q}) : \text{for some } \alpha \in \widehat{F_1} \text{ we have } \sigma\lambda = \alpha\lambda \right\}.$$

Remark. $\mathrm{Galr}(\lambda)$ is a subgroup of $\mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$ which can easily be computed from λ. The r in Galr stands for *restriction* to the special linear group. Obviously, $\mathrm{Galr}(\lambda)$ contains $\mathrm{Galg}(\lambda)$.

LEMMA 3.2. *We have defined an action of $\widehat{F_1}$ on $\mathscr{F}$ that preserves degrees. Furthermore, if $\lambda \in \mathscr{F}$ and $\alpha, \beta \in \widehat{F_1}$, then $\alpha\lambda = \beta\lambda$ if and only if* $\mathrm{Res}^{F_1}_{\mathscr{I}(\lambda)}(\alpha) = \mathrm{Res}^{F_1}_{\mathscr{I}(\lambda)}(\beta)$.

Proof. It is straightforward to check that we have an action of $\widehat{F_1}$ on $\mathscr{F}$. For the second part, it is enough to check that $\alpha\lambda = \lambda$ if and only if $\ker(\alpha) \supseteq \mathscr{I}(\lambda)$. In other words, we need to check that the stabilizer S of λ under the action of $\widehat{F_1}$ is exactly the set of all elements of $\widehat{F_1}$ whose kernel contains $\mathscr{I}(\lambda)$. Since F_1 is cyclic, we let γ be a generator of S. The set of elements of $\widehat{F_1}$ whose kernel contains $\ker(\gamma)$ is simply $\langle \gamma \rangle = S$. Hence, $\mathscr{I}(\lambda) = \ker(\gamma)$. The lemma follows.

THEOREM 3.3. *For each $\alpha \in \widehat{F_1}$ and each $\lambda \in \mathscr{F}_n$, $\alpha\chi_\lambda$ is an irreducible character of* $\mathbf{GL}(n, q)$, *and in fact*

$$\alpha\chi_\lambda = \chi_{\alpha\lambda}.$$

Proof. This is the content of the proposition in Section 3 of [9].

PROPOSITION 3.4. *Let $\lambda \in \mathscr{F}_n$. Then, the field of values of* $\mathrm{Res}^{\mathbf{GL}(n,q)}_{\mathbf{SL}(n,q)}(\chi_\lambda)$ *is*

$$\mathbf{Q}\left(\mathrm{Res}^{\mathbf{GL}(n,q)}_{\mathbf{SL}(n,q)}(\chi_\lambda)\right) = \mathbf{Q}(\lambda)^{\mathrm{Galr}(\lambda)} = \mathrm{Galr}(\lambda)',$$

in other words, it is simply the fixed field of $\mathrm{Galr}(\lambda)$.

Proof. Since $\mathbf{Q}(\lambda)/\mathbf{Q}$ is a finite Galois extension and $\mathbf{Q}(\chi_\lambda) \subseteq \mathbf{Q}(\lambda)$, we need to show that

$$\mathrm{Gal}\left(\mathbf{Q}(\lambda)/\mathbf{Q}\left(\mathrm{Res}^{\mathbf{GL}(n,q)}_{\mathbf{SL}(n,q)}(\chi_\lambda)\right)\right) = \mathrm{Galr}(\lambda).$$

If $\sigma \in \mathrm{Galr}(\lambda)$, as $\alpha \in \widehat{F_1}$ when considered as a linear character of $\mathbf{GL}(n, q)$ contains $\mathbf{SL}(n, q)$ in its kernel, then $\sigma \mathrm{Res}^{\mathbf{GL}(n,q)}_{\mathbf{SL}(n,q)}(\chi_\lambda) = \mathrm{Res}^{\mathbf{GL}(n,q)}_{\mathbf{SL}(n,q)}(\chi_\lambda)$, and it follows that

$$\sigma \in \mathrm{Gal}\left(\mathbf{Q}(\lambda)/\mathbf{Q}\left(\mathrm{Res}^{\mathbf{GL}(n,q)}_{\mathbf{SL}(n,q)}(\chi_\lambda)\right)\right).$$

Conversely, if $\sigma \in \mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q}(\mathrm{Res}_{\mathbf{SL}(n,q)}^{\mathbf{GL}(n,q)}(\chi_\lambda)))$, then the restriction of $\sigma\chi_\lambda$ to $\mathbf{SL}(n,q)$ equals the restriction of χ_λ to $\mathbf{SL}(n,q)$, whence, by Clifford theory, there exists some linear character α of $\mathbf{GL}(n,q)/\mathbf{SL}(n,q)$ such that $\sigma\chi_\lambda = \alpha\chi_\lambda$. Hence, $\sigma \in \mathrm{Galr}(\lambda)$, and we are done.

4. CHARACTERS OF $\mathbf{SL}(n,q)$ AND THEIR FIELDS OF VALUES

The characters of $\mathbf{SL}(n,q)$ have been parameterized by Lehrer [10]. In this section, we set some notation for this parameterization, and we calculate for each irreducible character its inertia group and its field of values.

DEFINITION 4.1. Let $\lambda \in \mathscr{F}_n$, then we denote by ψ_λ any irreducible character contained in $\mathrm{Res}_{\mathbf{SL}(n,q)}^{\mathbf{GL}(n,q)}(\chi_\lambda)$.

Remarks. The character ψ_λ is only defined up to conjugation by some element of $\mathbf{GL}(n,q)$. Our purpose here is to describe the field of values and the Schur indices of ψ_λ over all fields. As these are invariant under conjugation by elements $\mathbf{GL}(n,q)$, it does not matter, for our purposes, which irreducible of $\mathbf{SL}(n,q)$ is represented by ψ_λ.

PROPOSITION 4.2. *Each irreducible character of* $\mathbf{SL}(n,q)$ *is* $\mathbf{GL}(n,q)$ *conjugate to some* ψ_λ, *for some* $\lambda \in \mathscr{F}_n$. *Furthermore, if* $\lambda, \lambda' \in \mathscr{F}_n$, *then the* $\mathbf{GL}(n,q)$*-orbit of characters conjugate to* $\psi_{\lambda'}$ *is the same as that of characters conjugate to* ψ_λ *if and only if there exists some* $\alpha \in \widehat{F_1}$ *such that* $\lambda' = \alpha\lambda$.

Proof. This follows from Clifford Theory, Theorem 2.4, and Theorem 3.3.

PROPOSITION 4.3. *Let* $\lambda \in \mathscr{F}$. *Then, the inertia group in* $\mathbf{GL}(n,q)$ *of* ψ_λ *is the set of all elements of* $\mathbf{GL}(n,q)$ *whose determinant is in* $\mathscr{I}(\lambda)$; *see Definition* 3.1(5) *above. Furthermore, the set of all* $\alpha \in \widehat{F_1}$ *such that* $\alpha\lambda = \lambda$ *is the set of all elements of* $\widehat{F_1}$ *whose kernel contains* $\mathscr{I}(\lambda)$.

Proof. Let I be the inertia group of ψ_λ. By Clifford Theory, there exists an irreducible character ρ of I, with the properties that $\mathrm{Res}_{\mathbf{SL}(n,q)}^{I}(\rho) = \psi_\lambda$ and $\mathrm{Ind}_I^{\mathbf{GL}(n,q)}(\rho) = \chi_\lambda$. If $\alpha \in \widehat{F_1}$, then $\alpha\chi_\lambda = \chi_\lambda$ if and only if $\mathrm{Res}_I^{F_1}(\alpha)\rho = \rho$ if and only if $\ker(\alpha\,\mathbf{det}) \supseteq I$. Hence,

$$I = \bigcap \left\{\ker(\alpha\,\mathbf{det}) : \alpha \in \widehat{F_1} \text{ and } \alpha\chi_\lambda = \chi_\lambda\right\}.$$

By Theorem 2.4 and Theorem 3.3,

$$\left\{\alpha \in \widehat{F_1} : \alpha\chi_\lambda = \chi_\lambda\right\} = \left\{\alpha \in \widehat{F_1} : \alpha\lambda = \lambda\right\}.$$

By Lemma 3.2,

$$\bigcap \left\{\ker(\alpha) : \alpha \in \widehat{F_1} \text{ and } \alpha\lambda = \lambda\right\} = \mathscr{I}(\lambda).$$

The proposition follows.

COROLLARY 4.4. *Let* $\lambda \in \mathscr{F}_n$. *Then, the index* $[F_1 : \mathscr{I}(\lambda)]$ *divides* $(q-1, n)$.

Proof. Since $|F_1| = q - 1$, we only need to show that the index $[F_1 : \mathscr{I}(\lambda)]$ divides n. Let $\alpha \in \widehat{F_1}$ be a generator of the cyclic group of all elements of $\widehat{F_1}$ whose kernel contains $\mathscr{I}(\lambda)$. Then, the order of α is $[F_1 : \mathscr{I}(\lambda)]$. By Proposition 4.3, we have $\alpha\lambda = \lambda$. Hence, the values of λ are constant within each orbit of α on $\mathscr{G}$. Since the orbits of α all have $[F_1 : \mathscr{I}(\lambda)]$ elements of $\mathscr{G}$ in them, it follows from Definition 2.3 that $\mathbf{deg}(\lambda) = n$ is a multiple of $[F_1 : \mathscr{I}(\lambda)]$, as desired.

Remark. The corollary can also be obtained easily from the remark that the center of $\mathbf{GL}(n, q)$ will be contained in the inertia group of ψ_λ.

THEOREM 4.5 (Zelevinsky). *Let* $\lambda \in \mathscr{F}_n$. *View* $\mathbf{GL}(n, q)$ *as a set of matrices with coefficients in* $\mathbf{F}_q$. *Let* U *be the Sylow* p*-subgroup of* $\mathbf{GL}(n, q)$ *of unipotent upper triangular matrices. Then there exists a linear character* $\theta = \theta_\lambda$ *of* U *such that* $(\mathrm{Res}_U^{\mathbf{GL}(n,q)}(\chi_\lambda), \theta)_U = 1$. *Furthermore,* θ *has the following form: There is an additive character* $\alpha\colon \mathbf{F}_q \to \mathbf{C}^\times$ *and integers* $k_1, \dots, k_r$ *such that, for each matrix* $(u_{ij}) \in U$,

$$\theta\big((u_{ij})\big) = \alpha\Big(\sum u_{i,i+1}\Big),$$

where the sum is for $i = 1, \dots, n-1$ *but omits* $i = k_1, \dots, k_r$. *Furthermore,* $\theta = 1$ *if and only if* $\lambda(\rho)$ *is the partition* (n) *for some* $\rho \in \widehat{F_1}$.

Proof. See Proposition 12.4 in Zelevinsky [18].

LEMMA 4.6. *Keep the notation of Theorem* 4.5. *Let* $\nu \in F_1$ *be an element of order* $p-1$, *and let* $x \in \mathbf{GL}(n, q)$ *be the diagonal matrix whose entries are* $\nu^{n-1}, \nu^{n-2}, \dots, 1$. *Then* $\mathbf{Q}(\theta)$ *is contained in the field of pth roots of* 1, *the element* x *normalizes* U, *its determinant is* $\mathbf{det}(x) = \nu^{\binom{n}{2}}$, *and* $\theta^x = \tau\theta$, *where* τ *is a generator of the group* $\mathrm{Gal}(\mathbf{Q}(\theta)/\mathbf{Q})$.

Proof. It follows from the definition of θ that $\mathbf{Q}(\theta)$ is contained in the field of pth roots of 1. A direct computation shows that x normalizes U and that, for each $u = (u_{ij}) \in U$, the $(i, i+1)$th entry of xux^{-1} is $\nu u_{i,i+1}$. The element ν is a generator of the multiplicative group of the prime field $\mathbf{F}_p$ and can be thought of as the reduction modulo p of some integer b, where b is relatively prime to p. Hence, for each $u \in U$ we have

$\theta^x(u) = \theta(u)^b$. Setting τ to be the Galois automorphism of $\mathbf{Q}(\theta)$ which raises each pth root of 1 to its bth power, we see that $\tau\theta = \theta^x$. Since ν has order $p-1$ modulo p, τ is actually a generator of $\mathrm{Gal}(\mathbf{Q}(\theta)/\mathbf{Q})$ Finally, the computation of the determinant of x is straightforward.

LEMMA 4.7. *Keep the notation of Theorem* 4.5 *and Lemma* 4.6. *Let I be the inertia group of ψ_λ. If p is odd, q is not a square, $2 \le n_2 \le (p-1)_2$, and, for any element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$, then $\langle x \rangle \cap I$ has index 2 in $\langle x \rangle$. Otherwise, $\langle x \rangle \subseteq I$.*

Proof. By Proposition 4.3,

$$[\langle x \rangle : \langle x \rangle \cap I] = [\langle \mathbf{det}(x) \rangle : \langle \mathbf{det}(x) \rangle \cap \mathscr{I}(\lambda)].$$

Since F_1 is cyclic, $[\langle \mathbf{det}(x) \rangle : \langle \mathbf{det}(x) \rangle \cap \mathscr{I}(\lambda)]$ is the denominator of the fraction

$$f := \frac{|\mathscr{I}(\lambda)|}{|\langle \mathbf{det}(x) \rangle|}.$$

By Lemma 4.6, $\mathbf{det}(x) = \nu^{\binom{n}{2}}$. Hence, $|\langle \mathbf{det}(x) \rangle| = (p-1)/\gcd(p-1, \binom{n}{2})$. By Corollary 4.4, $|\mathscr{I}(\lambda)| = (q-1)/c$, where c is some integer dividing $(q-1, n)$. Hence

$$f = \frac{|\mathscr{I}(\lambda)|}{|\langle \mathbf{det}(x) \rangle|} = \frac{q-1}{p-1} \cdot \frac{\gcd\left(p-1, \binom{n}{2}\right)}{c}.$$

If r is an odd prime, the r-part f_r of f is an integer multiple of the r-part of $((q-1)/(p-1)) \cdot (\gcd(p-1, n)/\gcd(q-1, n))$, which is itself an integer. Hence, the denominator of f is a power of 2. The fraction $(q-1)/(p-1)$ is always an integer. If n is odd or $p = 2$, then the 2-part of $\gcd(p-1, \binom{n}{2})/c$ is an integer, so that f is an integer and the lemma holds in this case. Hence, we assume that n is even and p is odd. Now, the denominator of f is the 2-part of the denominator of

$$\frac{q-1}{p-1} \cdot \frac{\gcd\left(p-1, \dfrac{n}{2}\right)}{c}.$$

If $n_2 > (p-1)_2$, then the 2-part of f is an integer multiple of the 2-part of $((q-1)/(p-1)) \cdot (\gcd(p-1, n)/\gcd(q-1, n))$. Hence, the denominator of f is again 1 and the lemma holds in this case. Hence, we assume that $2 \le n_2 \le (p-1)_2$. Now, the 2-part of $\gcd(p-1, n/2)$ is $n_2/2$ and the 2-part of $\gcd(q-1,n)$ is n_2. If q is a square, then $(q-1)/(p-1)$ is an even integer, so that the denominator of f is 1 and the lemma holds in this case. We assume henceforth that q is not a square. Now $(q-1)/(p-1)$ is an odd integer. Hence, the denominator of f is 2 if $c_2 = n_2$, and 1 otherwise. By Proposition 4.3, $c_2 = n_2$ if and only if $\beta\lambda = \lambda$. Hence, the lemma holds.

THEOREM 4.8. *Let $\lambda \in \mathscr{F}_n$. Then the field of values $\mathbf{Q}(\psi_\lambda)$ of $\psi_\lambda \in \mathbf{Irr}(\mathbf{SL}(n,q))$ is as follows. If p is odd, q is not a square, $2 \le n_2 \le (p-1)_2$, and, for any element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$, then $\mathbf{Q}(\psi_\lambda) = \mathrm{Galr}(\lambda)'(\sqrt{\epsilon p})$, where $\epsilon \in \{1,-1\}$ and $p \equiv \epsilon \pmod 4$. Otherwise, $\mathbf{Q}(\psi_\lambda) = \mathrm{Galr}(\lambda)'$.*

Proof. We use the notation of Theorem 4.5, Lemma 4.6, and Lemma 4.7. Since $U \subseteq \mathbf{SL}(n,q)$, θ is contained in the restriction of exactly one of the $\mathbf{GL}(n,q)$ conjugates of ψ_λ, and we assume without loss that θ is contained in the restriction to U of ψ_λ. Let I be the inertia group of ψ_λ in $\mathbf{GL}(n,q)$. If $y \in \langle x\rangle \cap I$, then θ^y is contained in $\mathrm{Res}_U^{\mathbf{SL}(n,q)}(\psi_\lambda)$. Conversely, suppose $y \in \langle x\rangle$ and θ^y is contained in $\mathrm{Res}_U^{\mathbf{SL}(n,q)}(\psi_\lambda)$. Then there is a unique irreducible summand of $\mathrm{Res}_{\mathbf{SL}(n,q)}^{\mathbf{GL}(n,q)}(\chi_\lambda)$ which contains θ^y in its restriction to U. This irreducible character is both ψ_λ and ψ_λ^y. It follows that $y \in \langle x\rangle \cap I$. Hence, for $y \in \langle x\rangle$, we have that $\theta^y \subseteq \mathrm{Res}_U^{\mathbf{SL}(n,q)}(\psi_\lambda)$ if and only if $y \in \langle x\rangle \cap I$.

Set $F = \mathrm{Galr}(\lambda)'$, $K = F(\theta)$, and $\overline{K}$ to be the algebraic closure of K. Since $\mathrm{Res}_{\mathbf{SL}(n,q)}^{\mathbf{GL}(n,q)}(\chi_\lambda)$ is a sum of conjugates of ψ_λ, by Proposition 3.4, $F \subseteq \mathbf{Q}(\psi_\lambda)$. If $\sigma \in \mathrm{Gal}(\overline{K}/K)$, then, by Proposition 3.4, $\sigma\,\mathrm{Res}_{\mathbf{SL}(n,q)}^{\mathbf{GL}(n,q)}(\chi_\lambda) = \mathrm{Res}_{\mathbf{SL}(n,q)}^{\mathbf{GL}(n,q)}(\chi_\lambda)$, and as ψ_λ is the unique irreducible in $\mathrm{Res}_{\mathbf{SL}(n,q)}^{\mathbf{GL}(n,q)}(\chi_\lambda)$ which contains θ, it follows that $\sigma\psi_\lambda = \psi_\lambda$. Hence, $F \subseteq \mathbf{Q}(\psi_\lambda) \subseteq K$.

Suppose first that $\theta = 1$. Then, $\mathbf{Q}(\psi_\lambda) = F$ and, by Theorem 4.5, $\lambda(\rho)$ is the partition (n) for some $\rho \in \widehat{F_1}$. It follows that if $\beta \in \widehat{F_1}$ and $\beta\lambda = \lambda$ then $\beta = 1$. Hence, the theorem holds in this case.

Assume, henceforth, that $\theta \ne 1$. Let η_p be a primitive pth root of 1. Then $\mathbf{Q}(\theta) = \mathbf{Q}(\eta_p)$ and $K = F(\eta_p)$. Since F is contained in a field of p'th roots of 1, we have $\mathbf{Q}(\eta_p) \cap F = \mathbf{Q}$. Let τ_0 be the unique element of $\mathrm{Gal}(K/F)$ which restricts to $\tau \in \mathrm{Gal}(\mathbf{Q}(\theta)/\mathbf{Q})$ (see Lemma 4.6). We define the subgroup T of $\mathrm{Gal}(K/F)$ as follows: If p is odd, q is not a square, $2 \le n_2 \le (p-1)_2$ and, for any element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$, then $T = \langle \tau_0^2 \rangle$. Otherwise, we set $T = \langle \tau_0 \rangle = \mathrm{Gal}(K/F)$.

We now show that $\mathrm{Gal}(K/\mathbf{Q}(\psi_\lambda)) = T$. Suppose $\sigma \in T$. Then, by Lemma 4.6 and Lemma 4.7, there is some $y \in \langle x\rangle \cap I$ such that $\sigma\theta = \theta^y$. It follows that $\sigma\theta \subseteq \mathrm{Res}_U^{\mathbf{SL}(n,q)}(\psi_\lambda)$. Hence, $\sigma^{-1}\psi_\lambda$ is an irreducible character contained in $\mathrm{Res}_{\mathbf{SL}(n,q)}^{\mathbf{GL}(n,q)}(\chi_\lambda)$ which contains θ in its restriction to U. Hence, $\sigma^{-1}\psi_\lambda = \psi_\lambda$. Hence, $T \subseteq \mathrm{Gal}(K/\mathbf{Q}(\psi_\lambda))$. Conversely, suppose $\sigma \in$

$\mathrm{Gal}(K/\mathbf{Q}(\psi_\lambda))$. Then, $\sigma = \tau_0^m$ for some appropriate integer m. Now $\sigma\theta = \theta^{x^m}$ is contained in the restriction of ψ_λ to U. It follows that $x^m \in \langle x \rangle \cap I$. Hence, again by Lemma 4.6 and Lemma 4.7, we have that $\sigma \in T$. Hence, $\mathrm{Gal}(K/\mathbf{Q}(\psi_\lambda)) = T$.

Suppose p is odd, q is not a square, $2 \leq n_2 \leq (p-1)_2$, and, for any element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$. Then $\sqrt{\epsilon p} \in K$, where $\epsilon \in \{1, -1\}$ and $p \equiv \epsilon \pmod 4$, and, for $\sigma \in \mathrm{Gal}(K/\mathbf{Q}(\psi_\lambda))$, we have $\sigma \in T$ if and only if $\sigma(\sqrt{\epsilon p}) = \sqrt{\epsilon p}$. Hence, $\mathbf{Q}(\psi_\lambda) = F(\sqrt{\epsilon p})$, and the theorem holds in this case. Otherwise, $\mathbf{Q}(\psi_\lambda) = F$, and again the theorem holds. This concludes the proof of the theorem.

5. THE CENTRALIZER ALGEBRA FOR EACH IRREDUCIBLE CHARACTER

In this section we calculate the element of the Brauer group $[\psi_\lambda] \in \mathrm{Br}(\mathbf{Q}(\psi_\lambda))$ associated with each irreducible character $\psi_\lambda \in \mathbf{Irr}(\mathbf{SL}(n,q))$ in terms of cross products. Our main tool is the formula for calculating such elements in terms of cross products which was proved in [17].

Notation 5.1. Given K/F a finite Galois extension of degree n, with Galois group $\mathrm{Gal}(K/F) = \langle \tau \rangle$ cyclic and generated by τ and some element $a \in F^\times$, then there exists the cross product of K and τ with respect to a; see for example [13]. This is a central simple algebra over F of dimension n^2, containing K and a certain invertible element t which acts by conjugation on K as τ and such that $t^n = a$. Since we write our maps on the left, our convention is that $tkt^{-1} = \tau(k)$ for all $k \in K$. These conditions characterize the cross product up to an F-algebra isomorphism. We denote by $[K/F, \tau, a]$ the element of the Brauer group $\mathrm{Br}(F)$ which has the cross product of K and τ with respect to a as a representative.

THEOREM 5.2. *Let G be a finite group, $\chi \in \mathbf{Irr}(G)$, and F be a field containing $\mathbf{Q}(\chi)$. Let U be a subgroup of G and $\theta \in \mathbf{Irr}(U)$ be such that $\theta(1) = 1$ and $(\mathrm{Res}_U^G(\chi), \theta) = 1$. Set $K = F(\theta)$ and set $n = |\mathrm{Gal}(K/F)|$, and assume that $\mathrm{Gal}(K/F) = \langle \sigma \rangle$ is cyclic. Assume that there exist a group H containing G as a normal subgroup and a character $\bar{\chi} \in \mathbf{Irr}(H)$ extending the character χ, such that $F(\bar{\chi}) \cap K = F$. Assume, furthermore, that there is some element $x \in \mathrm{N}_H(U)$ such that, for all $u \in U$, $\theta(x^{-1}ux) = \sigma\theta(u)$ and $x^n \in U$. Then, there exists some h in the coset $x^{-1}G$ such that $\bar{\chi}(h) \neq 0$ and, for each such h, we have*

$$[\chi] = \left[K/F, \sigma, \bar{\chi}(h)^n \theta(x^n)\right].$$

Proof. This is Theorem 2.4 in [17].

LEMMA 5.3. *Let* $\lambda \in \mathscr{F}_n$, *and suppose* $t \in \mathscr{I}(\lambda)$. *Then there exists some* $x \in \mathbf{GL}(n, q)$ *such that* $\mathbf{det}(x) = t$ *and* $\chi_\lambda(x) \neq 0$.

Proof. Suppose $\chi_\lambda(x) = 0$ for all $x \in \mathbf{GL}(n, q)$ with $\mathbf{det}(x) = t$. Let κ be the characteristic function of the set $\{t\}$ in $\mathscr{I}(\lambda)$. Then, we may write κ as a linear combination $\sum_{\theta \in \mathbf{Irr}(\mathscr{I}(\lambda))} \mu_\theta \theta$, where $\mu_\theta \in \mathbf{C}$ and not all are 0. Each θ is a linear character, so we may extend it to a linear character $\theta_0 \in \widehat{F_1}$. Let $\kappa_0 = \sum_{\theta \in \mathbf{Irr}(\mathscr{I}(\lambda))} \mu_\theta \theta_0$. Hence, $\mathrm{Res}^{F_1}_{\mathscr{I}(\lambda)}(\kappa_0) = \kappa$. Let I be the preimage in $\mathbf{GL}(n, q)$ of $\mathscr{I}(\lambda)$. By Proposition 4.3, χ_λ is induced from some character ψ_0 of I. Since I is a normal subgroup of $\mathbf{GL}(n, q)$, χ_λ vanishes outside I. Hence, we have that $\kappa_0 \chi_\lambda = 0$. It follows that

$$0 = \kappa_0 \chi_\lambda = \sum_{\theta \in \mathbf{Irr}(\mathscr{I}(\lambda))} \mu_\theta \theta_0 \, \mathrm{Ind}_I^{\mathbf{GL}(n,q)}(\psi_0)$$

$$= \sum_{\theta \in \mathbf{Irr}(\mathscr{I}(\lambda))} \mu_\theta \, \mathrm{Ind}_I^{\mathbf{GL}(n,q)}(\theta\psi_0).$$

However, by Proposition 3.2, each $\mathrm{Ind}_I^{\mathbf{GL}(n,q)}(\theta\psi_0) = \theta_0 \chi_\lambda = \chi_{\theta_0\lambda}$ is a distinct irreducible character of $\mathbf{GL}(n, q)$. Hence, the equality contradicts the linear independence of irreducible characters. This completes the proof of the lemma.

LEMMA 5.4. *Let* $\lambda \in \mathscr{F}_n$. *Let* ψ_λ *be any of the irreducible characters of* $\mathbf{SL}(n, q)$ *parameterized by* λ, *and set* $F = \mathbf{Q}(\psi_\lambda)$ *and* $K = F(\eta_p)$, *where* η_p *is a primitive pth root of* 1. *Assume that, for* $\rho \in \widehat{F_1}$, $\lambda(\rho)$ *is not the partition* (n) *or, equivalently, assume that* $\psi_\lambda \neq 1$. *Let* $\nu \in F_1$ *be of order* $p - 1$. *Set* $\nu_0 = \nu^{n(n-1)}$ *and* $k = (p-1)/2$ *if p is odd, q is not a square,* $2 \leq n_2 \leq (p-1)_2$, *and, for any element* $\beta \in \widehat{F_1}$ *of order* n_2, *we have* $\beta\lambda = \lambda$; *set* $\nu_0 = \nu^{\binom{n}{2}}$ *and* $k = p - 1$ *otherwise. Then, the following hold*:

(1) *There exists some* $y \in \mathbf{GL}(n, q)$ *such that* $\mathbf{det}(y) = \nu_0$ *and* $\chi_\lambda(y) \neq 0$.

(2) *For each y satisfying the conditions of* (1), *we have that* $\chi_\lambda(y)^k \in F^\times$ *and, for some appropriate generator* σ *of* $\mathrm{Gal}(K/F)$, *we have*

$$[\psi_\lambda] = \left[K/F, \sigma, \chi_\lambda(y)^k\right].$$

Proof. Applying Theorem 4.5, we obtain a subgroup U and a linear character θ of U with the properties specified in the theorem. As ψ_λ is contained in $\mathrm{Res}^{\mathbf{GL}(n,q)}_{\mathbf{SL}(n,q)}(\chi_\lambda)$, and conjugation by elements of $\mathbf{GL}(n, q)$ does not affect the Schur index of the character, we assume without loss that $(\mathrm{Res}^{\mathbf{SL}(n,q)}_U(\psi_\lambda), \theta) = 1$. From our general assumption on λ and the properties specified in Theorem 4.5 it follows that $\theta \neq 1$. Furthermore, we keep the notation of Lemma 4.6 and Lemma 4.7. In particular, there is an element $x \in \mathbf{GL}(n,q)$ normalizing U such that $\theta^x = \tau\theta$, where τ is a generator of $\mathrm{Gal}(\mathbf{Q}(\eta_p)/\mathbf{Q})$. Setting $F_0 = \mathrm{Galr}(\lambda)'$, since $F_0 \cap \mathbf{Q}(\eta_p) = \mathbf{Q}$, we may extend τ uniquely to some element $\tau_0 \in \mathrm{Gal}(F_0(\eta_p)/F_0)$. We set $\sigma = \tau_0^2$ if p is odd, q is not a square, $2 \le n_2 \le (p-1)_2$, and, for any element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$; we set $\sigma = \tau_0$ otherwise. Following the proof of Theorem 4.8, we see that $K = F_0(\eta_p)$, $k = [K:F]$, and $\langle \sigma \rangle = \mathrm{Gal}(K/F)$.

By Lemma 4.7, we have that $\nu_0 \in \mathscr{A}(\lambda)$. Hence, by Lemma 5.3, there exists some $y \in \mathbf{GL}(n,q)$ such that $\mathbf{det}(y) = \nu_0$ and $\chi_\lambda(y) \neq 0$. Hence, (1) holds. Let I be the inertia group of ψ_λ in $\mathbf{GL}(n,q)$. Then, by Clifford's Theorem, there exists a unique extension ρ of ψ_λ to I such that $\mathrm{Ind}_I^{\mathbf{GL}(n,q)}(\rho) = \chi_\lambda$. The field of values of ρ is $F(\rho) = F(\chi_\lambda)$. By Proposition 2.8, $\mathbf{Q}(\chi_\lambda) \subseteq \mathbf{Q}(\lambda)$. Since $\mathbf{Q}(\lambda) \cap \mathbf{Q}(\eta_p) = \mathbf{Q}$, it follows that $F(\rho) \cap K = F$. Hence, by Theorem 5.2, there exists some $z \in I$ such that $\mathbf{det}(z) = \nu_0$ and $\rho(z) \neq 0$ and $[\psi_\lambda] = [K/F, \sigma, \rho(z)^k]$.

Let y be any element satisfying (1). We now show that $\chi_\lambda(y)/\rho(z) \in F^\times$. Let $\tau \in \mathrm{Gal}(\overline{\mathbf{Q}}/F)$. Then $\tau\chi_\lambda = \alpha\chi_\lambda$, for some $\alpha \in \widehat{F_1}$. It follows that $\tau\chi_\lambda$ is induced from $\mathrm{Res}_I^{F_1}(\alpha)\rho$, and also from $\tau\rho$. By the uniqueness in Clifford's Theorem, it follows that $\tau\rho = \mathrm{Res}_I^{F_1}(\alpha)\rho$. Hence, $\tau(\chi_\lambda(y)) = \alpha(\nu_0)\chi_\lambda(y)$ and $\tau(\rho(z)) = \alpha(\nu_0)\rho(z)$. Hence, $\chi_\lambda(y)/\rho(z) \in F^\times$. Since the value of $[K/F, \sigma, \rho(z)^k]$ is not affected when we multiply $\rho(z)^k$ by the norm of some element of K, and the norm of any element of F is its kth power, we have that $[K/F, \sigma, \rho(z)^k] = [K/F, \sigma, \chi_\lambda(y)^k]$. This completes the proof of the lemma.

LEMMA 5.5. *Assume the notation of Lemma* 5.4. *Let* ν_2 *be the* 2-*part of* ν_0. *Then*, *the following hold*:

(1) *There exists some* $y \in \mathbf{GL}(n,q)$ *such that* $\mathbf{det}(y) = \nu_2$ *and* $\chi_\lambda(y) \neq 0$.

(2) *For each y satisfying the conditions of* (1), *we have that* $\chi_\lambda(y)^k \in F^\times$ *and*, *for some appropriate generator* σ *of* $\mathrm{Gal}(K/F)$, *we have*

$$[\psi_\lambda] = \left[K/F, \sigma, \chi_\lambda(y)^k\right].$$

Proof. Since $\nu_0 \in \mathscr{A}(\lambda)$, we have that $\nu_2 \in \mathscr{A}(\lambda)$. Hence, (1) follows immediately from Lemma 5.3. Let y satisfy the conditions of (1). Let $\mu \in F_1$ be such that $\nu = \nu_2\mu$. Then μ has odd order and, by the definition of ν_0, μ is either the nth power or the $\frac{n}{2}$th power of some element of odd order dividing $p-1$ of F_1. Hence, in all cases there exists some $\omega \in F_1$ such that $\omega^n = \mu$ and ω has odd order. Set $x = y(\omega 1) \in \mathbf{GL}(n,q)$ to be the product in $\mathbf{GL}(n,q)$ of y with ω times the identity.

Then, $\mathbf{det}(x) = \mathbf{det}(y)\omega^n = \nu_0$. Let ρ be the unique linear character contained in the restriction of χ_λ to $Z(\mathbf{GL}(n, q))$, when viewed as an element of $\widehat{F_1}$. Then, $\chi_\lambda(x) = \rho(\omega)\chi_\lambda(y)$. In particular, $\chi_\lambda(x) \neq 0$, and by applying Lemma 5.4 we get

$$[\psi_\lambda] = \left[K/F, \sigma, \rho(\omega)^k \chi_\lambda(y)^k\right].$$

Since k is divisible by the full $2'$-part of $p-1$ and ω has odd order dividing $p-1$, it follows that $\rho(\omega)^k = 1$, and the results follow.

LEMMA 5.6. *Assume the notation of Lemmas* 5.4 *and* 5.5. *Then, the following hold.*

(1) *If $p = 2$, or n is odd, or $n_2 > (p-1)_2$, then $[\psi_\lambda] = 1$.*

(2) *Suppose p is odd, $2 \leq n_2 \leq (p-1)_2$, and q is a square. The restriction of χ_λ to $Z(\mathbf{GL}(n, q)) \simeq F_1$ is a multiple of some irreducible linear character, which we denote by ρ. Let m be the order of ρ. Then, if $m_2 = (q-1)_2$ we have $[\psi_\lambda] = [K/F, \sigma, -1]$ as an element of* $\mathrm{Br}(\mathbf{Q}(\psi_\lambda))$, *and if $m_2 \neq (q-1)_2$ we have $[\psi_\lambda] = 1$.*

(3) *Suppose p is odd, $n_2 = (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$. Then $[\psi_\lambda] = 1$.*

(4) *Suppose p is odd, $2 \leq n_2 < (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$. The restriction of χ_λ to $Z(\mathbf{GL}(n, q)) \simeq F_1$ is a multiple of some irreducible linear character, which we denote by ρ. Let m be the order of ρ. Then, if $m_2 = (q-1)_2$ we have $[\psi_\lambda] = [K/F, \sigma, -1]$ as an element of* $\mathrm{Br}(\mathbf{Q}(\psi_\lambda))$, *and if $m_2 \neq (q-1)_2$ we have $[\psi_\lambda] = 1$.*

Proof. Suppose that, for some $\omega \in F_1$, we have $\omega^n = \nu_2$. Let $\mathrm{Res}^{\mathbf{GL}(n,q)}_{Z(\mathbf{GL}(n,q))}(\chi_\lambda) = d\rho$, where $\rho \in \widehat{F_1}$ and d is a positive integer. Then, by Lemma 5.5,

$$[\psi_\lambda] = \left[K/F, \sigma, \chi_\lambda(\omega 1)^k\right] = \left[K, F, \sigma, \rho(\omega)^k\right],$$

since d^k is a norm.

If $p = 2$ or n is odd, ω does exist because ν_2 has order a power of 2. If $n_2 > (p-1)_2$, then $\nu_2 = 1$, so again ω exists. In these cases $k = p-1$ and $\omega^k = 1$. Hence, if $p = 2$, n is odd, or $n_2 > (p-1)_2$, then $[\psi_\lambda] = 1$. Hence, (1) holds.

Suppose p is odd, and $n_2 > (p-1)_2$, and q is a square. In this case, ν_2 has order $2(p-1)_2/n_2$. Since q is a square, there exists some $\omega \in F_1$ of order $2(p-1)_2$ such that $\omega^n = \nu_2$. In this case, we have $k = p-1$ and $\omega^k = -1$. Hence, $[\psi_\lambda] = [K/F, \sigma, \rho(-1)]$. Hence, (2) holds.

Suppose p is odd, $n_2 = (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$. In this case, $\nu_0 = \nu^{n(n-1)}$ and $k = (p-1)/2$. Hence, $\nu_2 = 1$ and, taking $\omega = 1$, we see that $[\psi_\lambda] = 1$. Hence, (3) holds.

Finally, suppose p is odd, $2 \leq n_2 < (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$. In this case, $\nu_0 = \nu^{n(n-1)}$ and $k = (p-1)/2$. Hence, the order of ν_2 is $(p-1)_2/n_2 > 1$. Hence, we may take some $\omega \in F_1$ of order $(p-1)_2$ such that $\omega^n = \nu_2$. In this case, we have $\omega^k = -1$ and $[\psi_\lambda] = [K/F, \sigma, \rho(-1)]$. Hence, (4) holds. This completes the proof of this lemma.

To handle the case not covered by the previous lemma, we need to introduce further notation.

Definition 5.7. Let $\lambda \in \mathscr{F}_n$ and assume that p is odd, q is not a square, $2 \leq n_2 \leq (p-1)_2$, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$. We fix some element $t \in F_1$ of order $2(p-1)_2/n_2$. Then, we define a map

$$\delta\colon \mathrm{Galr}(\lambda) \to \mathbf{C}^\times$$

as follows. Let $\sigma \in \mathrm{Galr}(\lambda)$. Then, by Definition 3.1, there exists some $\alpha \in \widehat{F_1}$ such that $\sigma\lambda = \alpha\lambda$. We set $\delta(\sigma) = \alpha(t)$.

Proposition 5.8. *The map δ of Definition 5.7 is well defined. Furthermore, its values are $2(p-1)_2/n_2$th roots of unity in $\mathbf{Q}(\lambda)$, and δ is a crossed homomorphism; in other words, for all $\sigma, \tau \in \mathrm{Galr}(\lambda)$, we have $\delta(\sigma\tau) = \delta(\sigma)\sigma(\delta(\tau))$. In addition, if $x \in \mathbf{GL}(n, q)$ is such that $\mathbf{det}(x) = t$, then, for each $\sigma \in \mathrm{Galr}(\lambda)$, we have*

$$\sigma(\chi_\lambda(x)) = \delta(\sigma)\chi_\lambda(x).$$

Proof. By hypothesis, if $\beta \in \widehat{F_1}$ has order n_2, then $\beta\lambda \neq \lambda$. By Lemma 3.2, it follows that the kernel of β does not contain $\mathscr{I}(\lambda)$. Hence, the index of $\mathscr{I}(\lambda)$ in F_1 is not divisible by n_2. It follows that $t \in \mathscr{I}(\lambda)$. Now, for each $\sigma \in \mathrm{Galr}(\lambda)$, by Definition 3.1, there exists some $\alpha \in \widehat{F_1}$ such that $\sigma\lambda = \alpha\lambda$. By Lemma 3.2 $\mathrm{Res}^{F_1}_{\mathscr{I}(\lambda)}(\alpha)$ is uniquely determined by σ. Hence, $\delta(\sigma) = \alpha(t)$ is well defined. Since $\delta(\sigma)$ is the value of a linear character on t, it is a $(2(p-1)_2/n_2)$th root of unity. Furthermore, α is the quotient of some character in the support of λ by some Galois conjugate of itself.

Hence, all the values of α are in $\mathbf{Q}(\lambda)$, and, in particular, so is $\delta(\sigma)$. Let $\sigma, \tau \in \mathrm{Galr}(\lambda)$. Then, there exist $\alpha, \gamma \in \widehat{F_1}$ such that $\sigma\lambda = \alpha\lambda$ and $\tau\lambda = \gamma\lambda$. It follows that

$$\sigma\tau\lambda = \sigma \circ (\gamma\lambda) = (\sigma\gamma)\alpha\lambda.$$

Hence, we have $\delta(\sigma\tau) = \delta(\sigma)\sigma(\delta(\tau))$, and δ is a cross homomorphism.

Finally, let $x \in \mathbf{GL}(n, q)$ be such that $\mathbf{det}(x) = t$. By Lemma 2.7 and Theorem 3.3, we have

$$\sigma(\chi_\lambda(x)) = \alpha(t)\chi_\lambda(x),$$

completing the proof of the proposition.

Remark. Note that, by Lemma 5.3, an element $x \in \mathbf{GL}(n, q)$ such that $\mathbf{det}(x) = t$ and $\chi_\lambda(x) \neq 0$ does exist. Hence, we could also have defined δ in terms of any non-zero character value of χ_λ on the set of elements whose determinant is t, say, $\chi_\lambda(x) \neq 0$, by setting $\delta(\sigma) = \sigma(\chi_\lambda(x))/\chi_\lambda(x)$.

The following theorem describes the element of the Brauer group associated to each irreducible character of $\mathbf{SL}(n, q)$. Note that, in Theorem 4.8, we describe $\mathbf{Q}(\psi_\lambda)$ for each $\psi_\lambda \in \mathbf{Irr}(\mathbf{SL}(n, q))$. The next theorem gives $[\psi_\lambda] \in \mathrm{Br}(\mathbf{Q}(\psi_\lambda))$ for each $\psi_\lambda \in \mathbf{Irr}(\mathbf{SL}(n, q))$.

THEOREM 5.9. *Let $\lambda \in F_n$. Let ψ_λ be any of the irreducible characters of $\mathbf{SL}(n, q)$ parameterized by λ. Set $F = \mathbf{Q}(\psi_\lambda)$ and $K = F(\eta_p)$, where η_p is a primitive pth root of* 1. *We take τ to be an appropriate generator of* $\mathrm{Gal}(K/F)$. *Then, the following hold*:

(1) *If $p = 2$, or n is odd, or $n_2 > (p-1)_2$, then $[\psi_\lambda] = 1$.*

(2) *Suppose p is odd, $2 \leq n_2 \leq (p-1)_2$, and q is a square. The restriction of χ_λ to $\mathbf{Z}(\mathbf{GL}(n, q)) \simeq F_1$ is a multiple of some irreducible linear character, which we denote by ρ. Let m be the order of ρ. Then, if $m_2 = (q-1)_2$ we have $[\psi_\lambda] = [K/F, \tau, -1]$ as an element of* $\mathrm{Br}(\mathbf{Q}(\psi_\lambda))$, *and if $m_2 \neq (q-1)_2$ we have $[\psi_\lambda] = 1$.*

(3) *Suppose p is odd, $n_2 = (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$. Then $[\psi_\lambda] = 1$.*

(4) *Suppose p is odd, $2 \leq n_2 < (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$. The restriction of χ_λ to $\mathbf{Z}(\mathbf{GL}(n, q)) \simeq F_1$ is a multiple of some irreducible linear character, which we denote by ρ. Let m be the order of ρ. Then, if $m_2 = (q-1)_2$ we have $[\psi_\lambda] = [K/F, \tau, -1]$ as an element of* $\mathrm{Br}(\mathbf{Q}(\psi_\lambda))$, *and if $m_2 \neq (q-1)_2$ we have $[\psi_\lambda] = 1$.*

(5) *Suppose p is odd, $2 \leq n_2 \leq (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$. Then, there exists some element $\gamma \in \mathbf{Q}(\lambda)^{\times}$ such that, for all $\sigma \in \mathrm{Galr}(\lambda)$, we have*

$$\sigma(\gamma) = \delta(\sigma)\gamma.$$

Furthermore, for each such γ, we have that $\gamma^{p-1} \in \mathbf{Q}(\psi_\lambda)$ and $[\psi_\lambda] = [K/F, \tau, \gamma^{p-1}]$ as an element of $\mathrm{Br}(\mathbf{Q}(\psi_\lambda))$.

These five cases are mutually exclusive and cover all the irreducible characters of $\mathbf{SL}(n, q)$.

Proof. Assume first that, for some $\rho \in \widehat{F_1}$, $\lambda(\rho) = (n)$. This means that ψ_λ is the trivial character of $\mathbf{SL}(n, q)$. Of course, in this case $[\psi_\lambda] = 1$. We also are in Case (1), (2), or (5) of our theorem. If we are in Case (1), the theorem holds. If we are in Case (2) then, by Proposition 2.5, $m_2 < (q-1)_2$, which implies that the theorem also holds. So assume we are in Case (5). Set $\gamma' = \rho(t)$, using the t of Definition 5.7. It follows from Definition 5.7 that $\delta(\sigma) = \sigma(\gamma')/\gamma'$, for all $\sigma \in \mathrm{Galr}(\lambda)$. This implies that $\gamma'/\gamma \in \mathbf{Q}^{\times}$. Since γ' is a $(p-1)$st root of 1, γ^{p-1} is the $(p-1)$st power of a non-zero rational number. Hence, $[K/F, \tau, \gamma^{p-1}] = 1$ and the theorem holds in this case. We assume, henceforth, that for each $\rho \in \widehat{F_1}$, $\lambda(\rho) \neq (n)$.

Lemma 5.6 now tells us that (1)–(4) of our theorem hold. Hence, we only need to show the remaining case, that is, (5). Suppose p is odd, $2 \leq n_2 \leq (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$. Assume the notation of Lemma 5.4 and Lemma 5.5. In our case, we have $\nu_0 = \nu^{\binom{n}{2}}$ and $k = p-1$. The element ν_2 is the 2-part of ν_0, so ν_2 has order $2(p-1)_2/n_2$. We identify ν_2 with the element t of Definition 5.7. By Lemma 5.5, there exists some $y \in \mathbf{GL}(n, q)$ such that $\mathbf{det}(y) = t$, $\chi_\lambda(y) \neq 0$, and

$$[\psi_\lambda] = \left[K/F, \tau, \chi_\lambda(y)^{p-1}\right].$$

By Proposition 5.8, for each $\sigma \in \mathrm{Galr}(\lambda)$, we have

$$\sigma(\chi_\lambda(y)) = \delta(\sigma)\chi_\lambda(y).$$

Hence, an element with the properties listed in (5) for γ does exist. Let now $\gamma \in \mathbf{Q}(\lambda)^{\times}$ be any element such that, for all $\sigma \in \mathrm{Galr}(\lambda)$, we have

$$\sigma(\gamma) = \delta(\sigma)\gamma.$$

Then, $\chi_\lambda(y)/\gamma \in F^\times$ and $(\chi_\lambda(y)/\gamma)^{p-1}$ is its norm from K to F. It then follows that

$$\left[K/F, \tau, \chi_\lambda(y)^{p-1}\right] = \left[K/F, \tau, \gamma^{p-1}\right].$$

Hence, the theorem holds.

This theorem becomes simpler to state for the characters of $\mathbf{PSL}(n, q)$.

LEMMA 5.10. *Let $\lambda \in F_n$. Let ψ_λ be any of the irreducible characters of $\mathbf{SL}(n, q)$ parameterized by λ. Assume that the restriction of χ_λ to $\mathrm{Z}(\mathbf{GL}(n, q)) \simeq F_1$ is a multiple of some irreducible linear character, which we denote by $\rho \in \widehat{F_1}$. Let m be the order of ρ. Then ψ_λ can be viewed as a character of $\mathbf{PSL}(n, q)$ if and only if m divides $(q-1)/\mathrm{gcd}(q-1, n)$. Furthermore, suppose $\alpha \in \widehat{F_1}$. Then $\alpha\chi_\lambda = \chi_{\alpha\lambda}$ is an irreducible character of $\mathbf{GL}(n, q)$ and its restriction to $\mathrm{Z}(\mathbf{GL}(n, q))$ is a multiple of the linear character $\alpha^n\rho$.*

Proof. Multiplication of the identity of $\mathbf{GL}(n, q)$ by the scalars of the field $\mathbf{F}_q$ provides the isomorphism between F_1 and $\mathrm{Z}(\mathbf{GL}(n, q))$. The intersection $\mathrm{Z}(\mathbf{GL}(n, q)) \cap \mathbf{SL}(n, q)$ is the subgroup of $\mathrm{Z}(\mathbf{GL}(n, q))$ of order $\mathrm{gcd}(q-1, n)$, so it is in the kernel of ψ_λ if and only if m divides $(q-1)/\mathrm{gcd}(q-1, n)$. The determinant function on $\mathrm{Z}(\mathbf{GL}(n, q))$ translates into the map $F_1 \to F_1$ which raises every element to its nth power. Hence, the restriction of χ_λ is indeed a multiple of the irreducible character corresponding to $\alpha^n\rho$.

COROLLARY 5.11. *Let $\lambda \in F_n$. Let ψ_λ be any of the irreducible characters of $\mathbf{SL}(n, q)$ parameterized by λ. Set $F = \mathbf{Q}(\psi_\lambda)$ and $K = F(\eta_p)$, where η_p is a primitive pth root of 1. We take τ to be an appropriate generator of* $\mathrm{Gal}(K/F)$. *Assume, furthermore, that ψ_λ can be viewed as a character of $\mathbf{PSL}(n, q)$. Then, the following hold:*

(1) *If $p = 2$, n is odd, $n_2 > (p-1)_2$, q is a square, or, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$, then $[\psi_\lambda] = 1$.*

(2) *Suppose p is odd, $2 \le n_2 \le (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$. Then there exists some element $\gamma \in \mathbf{Q}(\lambda)^\times$ such that, for all $\sigma \in$ Galr(λ), we have*

$$\sigma(\gamma) = \delta(\sigma)\gamma.$$

Furthermore, for each such γ we have that $\gamma^{p-1} \in \mathbf{Q}(\psi_\lambda)$ and $[\psi_\lambda] = [K/F, \tau, \gamma^{p-1}]$ as an element of $\mathrm{Br}(\mathbf{Q}(\psi_\lambda))$.

Proof. Suppose $p = 2$, or n is odd, or $n_2 > (p-1)_2$, or q is a square, or, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$.

Then, we are in one of the cases (1)–(4) of Theorem 5.9. The corollary holds if we are in Case (1) or (3). If we are in Case (2) or (4), then, by Lemma 5.10, $m_2 \neq (q-1)_2$, so that $[\psi_\lambda] = 1$, and the corollary also holds in these cases. Hence, we need to assume that p is odd, $2 \leq n_2 \leq (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$. This implies that we are in Case (5) of Theorem 5.9. Hence, the corollary holds in all cases.

6. REAL SCHUR INDICES

In the previous section we saw that the element of the Brauer group $[\psi_\lambda] \in \mathrm{Br}(\mathbf{Q}(\psi_\lambda))$ associated to each $\psi_\lambda \in \mathbf{Irr}(\mathbf{SL}(n,q))$ can be expressed explicitly as a cross product of the form $[K/F, \tau, \gamma]$, where $F = \mathbf{Q}(\psi_\lambda)$, $K = F(\eta_p)$ (where η_p is a primitive pth root of 1), τ is a generator of $\mathrm{Gal}(K/F)$, and $\gamma \in F^\times$. The Schur index $\mathrm{m}_{\mathbf{Q}}(\psi_\lambda)$ of ψ_λ is then simply the order of this element of $\mathrm{Br}(F)$, and it is trivial if and only if γ is a norm from K.

For each p which is either ∞ or a finite prime, there is the local Schur index $\mathrm{m}_p(\psi_\lambda)$; see, for example, [3] for details. The local Schur index of ψ_λ at p is the order of the image of $[\psi_\lambda]$ in the Brauer group of the compositum of F and $\mathbf{Q}_p$ (the field of real numbers if $p = \infty$, and the field of p-adic numbers if p is a finite prime). Our goal is to calculate $\mathrm{m}_p(\psi_\lambda)$ for all p and all $\psi_\lambda \in \mathbf{Irr}(\mathbf{SL}(n,q))$. The (rational) Schur index of ψ_λ is then the lowest common multiple of all the local Schur indices $\mathrm{m}_p(\psi_\lambda)$, for p running through ∞ and all the finite primes.

In the present section, we calculate $\mathrm{m}_\infty(\psi_\lambda)$, the real Schur index of ψ_λ, for each irreducible character $\psi_\lambda \in \mathbf{Irr}(\mathbf{SL}(n,q))$. The character of order 2 of $\widehat{F_1}$ plays a special role in what follows, so we begin by fixing some notation for it.

Notation 6.1. Assume that p is odd. Then, we denote by **sgn** the element of $\widehat{F_1}$ of order 2.

Recall that, for each integer r which is relatively prime to m, we have set $\sigma_r \in \mathrm{Gal}(\mathbf{Q}(\eta_m)/\mathbf{Q})$ be the unique automorphism of $\mathbf{Q}(\eta_m)$ such that $\sigma_r(\eta_m) = \eta_m^r$.

PROPOSITION 6.2. *Let $\lambda \in F_n$. Let ψ_λ be any of the irreducible characters of $\mathbf{SL}(n,q)$ parameterized by λ. Then the following are equivalent*:

(1) *ψ_λ has real values.*

(2) *There exists some $\alpha \in \widehat{F_1}$ such that $\sigma_{-1}\lambda = \alpha\lambda$, and, furthermore, if p is odd, q is not a square, $2 \leq n_2 \leq (p-1)_2$, and, for any element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$, then $p \equiv 1 \pmod 4$.*

Proof. The field of values $\mathbf{Q}(\psi_\lambda)$ is described in Theorem 4.8. Note, that σ_{-1} is complex conjugation in $\mathbf{Q}(\lambda)$. Hence, $\mathrm{Galr}(\lambda)'$ is contained in $\mathbf{R}$ if and only if $\sigma_{-1} \in \mathrm{Galr}(\lambda)$. By Definition 3.1, $\sigma_{-1} \in \mathrm{Galr}(\lambda)$ if and only if there exists some $\alpha \in \widehat{F_1}$ such that $\sigma_{-1}\lambda = \alpha\lambda$. The proposition then follows immediately from Theorem 4.8.

COROLLARY 6.3. *Let $\lambda \in F_n$. Let ψ_λ be any of the irreducible characters of $\mathbf{SL}(n,q)$ parameterized by λ. Assume that ψ_λ has real values and $4 \le n_2 \le (q-1)_2$. Let the restriction of χ_λ to $\mathrm{Z}(\mathbf{GL}(n,q)) \simeq F_1$ be a multiple of $\rho \in \widehat{F_1}$. Let m be the order of ρ. Then, $m_2 \ne (q-1)_2$.*

Proof. Suppose $m_2 = (q-1)_2$. By Proposition 6.2 and Lemma 5.10 we have $\rho^{-1} = \sigma_{-1}\rho = \alpha^n\rho$, for some $\alpha \in \widehat{F_1}$. This means that $\rho^2 = \alpha^{-n}$. The 2-part of the order of ρ^2 is $(q-1)_2/2 \ge 2$, whereas the 2-part of the order of α^{-n} is at most $(q-1)_2/4$. This contradiction completes the proof of the corollary.

The following results describe the real Schur index of each irreducible character of $\mathbf{SL}(n,q)$. This Schur index is, of course, always either 1 or 2.

LEMMA 6.4. *Let $\lambda \in F_n$. Let ψ_λ be any of the irreducible characters of $\mathbf{SL}(n,q)$ parameterized by λ. Then, the real Schur index $\mathrm{m}_\infty(\psi_\lambda)$ is 2 if and only if all of the following hold*:

(1) *$n_2 = 2$, p is odd, and there exists some $\alpha \in \widehat{F_1}$ such that $\sigma_{-1}\lambda = \alpha\lambda$.*

(2) *Suppose q is a square. The restriction of χ_λ to $\mathrm{Z}(\mathbf{GL}(n,q)) \simeq F_1$ is a multiple of $\rho \in \widehat{F_1}$. Let m be the order of ρ. Then $m_2 = (q-1)_2$.*

(3) *Suppose q is not a square and we have $\mathbf{sgn}\,\lambda = \lambda$. The restriction of χ_λ to $\mathrm{Z}(\mathbf{GL}(n,q)) \simeq F_1$ is a multiple of $\rho \in \widehat{F_1}$. Let m be the order of ρ. Then, $p \equiv 1 \pmod 4$ and $m_2 = (q-1)_2$.*

(4) *If q is not a square and we have $\mathbf{sgn}\,\lambda \ne \lambda$, then the order of α is divisible by $(p-1)_2$.*

Proof. Suppose first that $\mathrm{m}_\infty(\psi_\lambda) = 2$. By Theorem 5.9(1), p is odd and $2 \le n_2 \le (p-1)_2$. Furthermore, ψ_λ has real values so that, by Proposition 6.2, there exists some $\alpha \in \widehat{F_1}$ such that $\sigma_{-1}\lambda = \alpha\lambda$. In addition, if q is not a square, and, for any element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$, then $p \equiv 1 \pmod 4$.

Suppose that $4 \le n_2$. Then, by Corollary 6.3, we have $m_2 \ne (q-1)_2$. It follows from Theorem 5.9 that, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$. Furthermore, there exists some element $\gamma \in \mathbf{Q}(\lambda)^{\times}$ such that, for all $\sigma \in \mathrm{Galr}(\lambda)$, we have

$$\sigma(\gamma) = \delta(\sigma)\gamma.$$

Furthermore, for each such γ, we have $\gamma^{p-1} \in \mathbf{Q}(\psi_\lambda)$ and $[\psi_\lambda] = [K/F, \tau, \gamma^{p-1}]$ as an element of $\mathrm{Br}(\mathbf{Q}(\psi_\lambda))$. Since $\mathrm{m}_\infty(\psi_\lambda) = 2$, we must have that γ^{p-1} is a negative real number. However, by Proposition 5.8, $\delta(\sigma_{-1})$ is a $(2(p-1)_2/n_2)$th root of 1. Hence, $(\delta(\sigma_{-1}))^{(p-1)/2} = 1$. Since $\sigma_{-1}(\gamma) = \delta(\sigma_{-1})\gamma$, it follows that $\gamma^{(p-1)/2} \in \mathbf{R}$, and this in turn implies that γ^{p-1} is positive. This contradiction shows that $n_2 = 2$. Furthermore, this argument also shows that if we are in Case (5) of Theorem 5.9, $\mathrm{m}_\infty(\psi_\lambda) = 1$, unless $(p-1)_2$ divides the order of $\delta(\sigma_{-1})$. By Definition 5.7, this implies that, in this case, the order of α is divisible by $(p-1)_2$. Hence, if $\mathrm{m}_\infty(\psi_\lambda) = 2$, it follows from Theorem 5.9 that (1)–(4) of our lemma hold.

Conversely, suppose that (1)–(4) of our lemma hold. By Proposition 6.2, ψ_λ has real values. We use the notation of Theorem 5.9. Note that F is a real field and K is not contained in $\mathbf{R}$. Hence, if $[\psi_\lambda] = [K/F, \tau, a]$ then $\mathrm{m}_\infty(\psi_\lambda) = 2$ if and only if $a < 0$. By Theorem 5.9, it follows immediately that $\mathrm{m}_\infty(\psi_\lambda) = 2$ if either q is a square or if q is not a square and $\mathbf{sgn}\,\lambda = \lambda$. Hence, we assume that q is not a square and $\mathbf{sgn}\,\lambda \neq \lambda$. Hence, by Theorem 5.9(5), there exists some element $\gamma \in \mathbf{Q}(\lambda)^{\times}$ such that, for all $\sigma \in \mathrm{Galr}(\lambda)$, we have

$$\sigma(\gamma) = \delta(\sigma)\gamma,$$

and, furthermore, $\gamma^{p-1} \in \mathbf{Q}(\psi_\lambda)$ and $[\psi_\lambda] = [K/F, \tau, \gamma^{p-1}]$ as an element of $\mathrm{Br}(F)$. By Definition 5.7, $\delta(\sigma_{-1}) = \alpha(t)$, where t is an element of order $(p-1)_2$. Hence, as the order of α is divisible by $(p-1)_2$, it follows that $\delta(\sigma_{-1})$ is a primitive $(p-1)_2$th root of 1. It follows that $\gamma^{(p-1)/2}$ is not a real number, for complex conjugation sends it to its negative. Hence, γ^{p-1} is a negative real number. Hence, $\mathrm{m}_\infty(\psi_\lambda) = 2$ in all cases. The lemma is proved.

THEOREM 6.5. *Let $\lambda \in F_n$. Let ψ_λ be any of the irreducible characters of $\mathbf{SL}(n,q)$ parameterized by λ. Then the real Schur index $\mathrm{m}_\infty(\psi_\lambda)$ is 2 if and only if all of the following hold*:

(1) *$n_2 = 2$, p is odd, and there exists some $\alpha \in \widehat{F_1}$ such that $\sigma_{-1}\lambda = \alpha\lambda$.*

(2) *Suppose $q \equiv 1 \pmod 4$. The restriction of χ_λ to $\mathbf{Z}(\mathbf{GL}(n,q)) \simeq F_1$ is a multiple of $\rho \in \widehat{F_1}$. Let m be the order of ρ. Then, $m_2 = (q-1)_2$.*

(3) *Suppose $q \equiv -1 \pmod 4$. Then $\mathbf{sgn}\,\lambda \neq \lambda$ and the order of α is even.*

Proof. By Lemma 6.4, we assume without loss that $n_2 = 2$, and p is odd, and there exists some $\alpha \in \widehat{F_1}$ such that $\sigma_{-1}\lambda = \alpha\lambda$, that is to say, we assume that (1) holds. If q is a square, then $q \equiv 1 \pmod 4$, so the result follows immediately from Lemma 6.4. Hence, we assume that q is not a square.

Suppose $p \equiv 1 \pmod 4$. Note that, since q is not a square, we have $(q-1)_2 = (p-1)_2$. The restriction of χ_λ to $\mathbf{Z}(\mathbf{GL}(n,q)) \simeq F_1$ is a multiple of $\rho \in \widehat{F_1}$. Let m be the order of ρ. By (1) and Lemma 5.10, restricting $\sigma_{-1}\chi_\lambda$ to the center of $\mathbf{GL}(n,q)$, we have $\rho^{-1} = \alpha^n\rho$. If $m_2 = (q-1)_2$, then the order of α^n is divisible by $(q-1)_2/2 > 1$, which implies that the order of α is divisible by $(q-1)_2$. Conversely, and similarly, if the order of α is divisible by $(q-1)_2$ then $m_2 = (q-1)_2$. It then follows immediately from Lemma 6.4 that the theorem holds if $q \equiv 1 \pmod 4$.

Finally, assume that q is not a square and $q \equiv -1 \pmod 4$. Then, if $\mathbf{sgn}\,\lambda = \lambda$, by (3) of Lemma 6.4, $\mathrm{m}_\infty(\psi_\lambda) = 1$, so that the theorem holds in this case. Hence, we assume that $\mathbf{sgn}\,\lambda \neq \lambda$. In this case, by Lemma 6.4, since $(p-1)_2 = 2$, we have that $\mathrm{m}_\infty(\psi_\lambda) = 2$ if and only if the order of α is even. This completes the proof of the theorem.

From the above theorem, we easily obtain the following, first proved by Gow [6], which describes the real Schur indices in the case $q \equiv 1 \pmod 4$.

COROLLARY 6.6. *Let $q \equiv 1 \pmod 4$ be a power of a prime, and let ψ be an irreducible character of $\mathbf{SL}(n,q)$, for some n. Then, $\mathrm{m}_\infty(\psi) = 2$ if and only if ψ is real valued, $n_2 = 2$, and ψ does not have the central involution of $\mathbf{SL}(n,q)$ in its kernel.*

Proof. By Theorem 6.5, the corollary holds if $n_2 \neq 2$. Hence, assume $n_2 = 2$. By Proposition 6.2, in this case, ψ has real values if and only if (1) of Theorem 6.5 holds. It is clear that the condition that ψ does not contain the central involution of $\mathbf{SL}(n,q)$ in its kernel is equivalent to (2) of Theorem 6.5. The corollary then follows immediately from Theorem 6.5.

Unlike what was speculated in [6], however, the situation is quite different in the case $q \equiv -1 \pmod 4$, as soon as $n \geq 6$. In the next corollary we show that the real Schur index is totally unrelated to whether or not the central involution is in the kernel for the set of irreducible characters of each $\mathbf{SL}(n,q)$, as soon as $q \equiv -1 \pmod 4$, $n_2 = 2$, and $n \geq 6$. A similar result was noted in [14] for the case $n = 6$.

COROLLARY 6.7. *Let $q \equiv -1 \pmod 4$ be a power of a prime and let $n > 2$ be an integer such that $n_2 = 2$. Then, there exist $\psi_1, \psi_2, \psi_3, \psi_4 \in \mathbf{Irr}(\mathbf{SL}(n,q))$ with the following properties:*

(1) *$\psi_1, \dots, \psi_4$ are all real valued.*

(2) *ψ_1 and ψ_2 can be thought of as characters of* $\mathbf{PSL}(n,q)$, *and ψ_3 and ψ_4 do not have the central involution of* $\mathbf{SL}(n,q)$ *in their kernels.*

(3) *ψ_1 and ψ_3 have real Schur index* 1, *and ψ_2 and ψ_4 have real Schur index* 2.

Proof. We may set ψ_1 to be the trivial character of $\mathbf{PSL}(n,q)$. We define $\lambda_3 \in \mathscr{F}_n$ as follows. We set $\lambda_3(\mathbf{sgn}) = (n-1)$ and $\lambda_3(1) = (1)$, where $(n-1)$ and (1) represent the partition with only one row of length respectively $n-1$ and 1. Furthermore, we set λ_3 to assign the empty partition to every other element of $\mathscr{G}$. Clearly, $\lambda_3 \in \mathscr{F}_n$, so we set $\psi_3 = \psi_{\lambda_3} \in \mathbf{Irr}(\mathbf{SL}(n,q))$. Since $n-1$ is odd, the central involution of $\mathbf{SL}(n,q)$ is not in the kernel of ψ_3; see Proposition 2.5. Furthermore, we have $\mathbf{sgn}\,\lambda_3 \neq \lambda_3$ (as $n > 2$) and $\sigma_{-1}\lambda_3 = \lambda_3$. By Proposition 6.2, it follows that ψ_3 has real values and, by Theorem 6.5, $\mathrm{m}_\infty(\psi_3) = 1$.

Let $\theta \in \widehat{F_2}$ have order $(q^2-1)_2$. Note that $\theta^q = \sigma_q\theta \neq \theta = \sigma_q^{\,2}\theta$ and that $\theta, \theta^q \in \mathscr{G}_2$. When viewed as an element of $\widehat{F_2}$ (see Definition 3.1), $\mathbf{sgn}$ is simply $\theta^{(q+1)_2}$, the element of order 2 of $\widehat{F_2}$. Hence, $\sigma_{-1}\theta = \theta^{-1} = \mathbf{sgn}\,\theta^q$ and $\sigma_{-1}\theta^q = \theta^{-q} = \mathbf{sgn}\,\theta$. Furthermore, $\mathbf{sgn}\,\theta \notin \{\theta, \theta^q\}$. Set $\iota = \theta^{(q+1)_2/2}$. Then, $\iota \in \mathscr{G}_2$ has order 4, and $\sigma_{-1}\iota = \mathbf{sgn}\,\iota = \sigma_q\iota \neq \iota$. Define $\lambda_2(\theta) = \lambda_2(\theta^q) = (1)$, and $\lambda_2(\iota) = \lambda_2(\mathbf{sgn}\,\iota) = ((n-4)/2)$, and $\lambda_2(1) = \lambda_2(\mathbf{sgn}) = (1)$. We let λ_2 assign the empty partition to all other elements of $\mathscr{G}$. Then, $\lambda_2 \in \mathscr{F}_n$, $\mathbf{sgn}\,\lambda_2 \neq \lambda_2$, and $\sigma_{-1}\lambda_2 = \mathbf{sgn}\,\lambda_2$. We set $\psi_2 = \psi_{\lambda_2}$. It then follows from Proposition 6.2 and Theorem 6.5 that ψ_2 has real values and $\mathrm{m}_\infty(\psi_2) = 2$. Furthermore, by Proposition 2.5, ψ_2 can be thought of as being a character of $\mathbf{PSL}(n,q)$.

Finally, we define λ_4 with $\lambda_4(\theta) = \lambda_4(\theta^q) = (1)$ and $\lambda_4(\iota) = \lambda_4(\mathbf{sgn}\,\iota) = ((n-2)/2)$, and by letting λ_4 assign the empty partition to all the other elements of $\mathscr{G}$. This time, we have that $\psi_4 = \psi_{\lambda_4} \in \mathbf{Irr}(\mathbf{SL}(n,q))$ has real values, $\mathrm{m}_\infty(\psi_4) = 2$, and the central involution of $\mathbf{SL}(n,q)$ is not in its kernel. This concludes the proof of the corollary.

7. p-LOCAL SCHUR INDICES

In this section, we describe the local Schur indices $\mathrm{m}_p(\psi_\lambda)$ for each irreducible character $\psi_\lambda \in \mathbf{Irr}(\mathbf{SL}(n,q))$, where p is the prime divisor of q.

LEMMA 7.1. *Let $\lambda \in \mathscr{F}_n$. Let F be a p-adic completion of the field of values $\mathbf{Q}(\psi_\lambda)$ of $\psi_\lambda \in \mathbf{Irr}(\mathbf{SL}(n,q))$. We let v_F be the discrete valuation of F normalized so that $v_F(F^\times) = \mathbf{Z}$. Let*

$$R = \{x \in F : v_F(x) \geq 0\}$$

be its valuation ring, let $\mathfrak{p} = \{x \in R : v_F(x) > 0\}$ *be its unique maximal ideal. If* p *is odd,* q *is not a square,* $2 \le n_2 \le (p-1)_2$, *and, for any element* $\beta \in \widehat{F_1}$ *of order* n_2, *we have* $\beta\lambda = \lambda$, *then we set* $d = (p-1)/2$, *and otherwise we set* $d = p - 1$. *Let* η_p *be a primitive pth root of* 1, *and let* $K = F(\eta_p)$. *Let* $a \in F^{\times}$, *and assume that* $v_F(a) = 0$. *Then* a *is a norm from* K *if and only if the projection* $\bar{a} \in (R/\mathfrak{p})^{\times}$ *of* a *is the dth power of some element of* $R/\mathfrak{p}$.

Proof. Since $\mathbf{Q}(\lambda)$ is an extension of $\mathbf{Q}$ by p'th roots of 1, p is unramified in the extension $\mathrm{Galr}(\lambda)'/\mathbf{Q}$. By Theorem 4.8, the field of values $\mathbf{Q}(\psi_\lambda)$ is as follows. If p is odd, q is not a square, $2 \le n_2 \le (p-1)_2$, and, for any element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$, then $\mathbf{Q}(\psi_\lambda) = \mathrm{Galr}(\lambda)'(\sqrt{\epsilon p})$, where $\epsilon \in \{1, -1\}$ and $p \equiv \epsilon \pmod 4$. Otherwise, $\mathbf{Q}(\psi_\lambda) = \mathrm{Galr}(\lambda)'$. This means that in all cases any prime divisor of p is totally ramified in the extension $\mathbf{Q}(\psi_\lambda)/\mathrm{Galr}(\lambda)'$. Furthermore, the extension $\mathbf{Q}(\psi_\lambda)(\eta_p)/\mathbf{Q}(\psi_\lambda)$ is totally ramified at any prime divisor of p and it has degree d.

The extension K/F is Galois, totally ramified, and its degree is d. We let v_K be the normalized valuation on K and let π_K be a uniformizer for v_K. We set π to be the norm in F of π_K. Then π is a uniformizer of F. Let S be the valuation ring of K and let $\mathfrak{p}_K$ be its maximal ideal. R is contained in S and the inclusion map generates an isomorphism $R/\mathfrak{p} \to S/\mathfrak{p}_K$. Let $b \in K^{\times}$. We write $b = b_0\pi_K^n$, where $b_0 \in K^{\times}$, $v_K(b_0) = 0$, and $n \in \mathbf{Z}$. Let c_0 be the norm of b_0. Since the extension is totally ramified, each element $\sigma \in \mathrm{Gal}(K/F)$ fixes each element of the residue field $S/\mathfrak{p}_K$. Hence, each element of $\mathrm{Gal}(K/F)$ fixes $\overline{b_0}$, the projection of b_0 into the residue field $S/\mathfrak{p}_K$ of K. Therefore, $\overline{c_0}$ is the dth power of some element of $S/\mathfrak{p}_K$. Since $c_0 \in F^{\times}$, we may also say that the projection of c_0 in $R/\mathfrak{p}$ is the dth power of some element of $R/\mathfrak{p}$. The norm of π_K is, by definition, π, so we have that the norm of b is an element of $F^{\times}$ of the form $c_0\pi^n$, where $n \in \mathbf{Z}$ and the projection of c_0 in $R/\mathfrak{p}$ is a dth power. The set of all such elements of $F^{\times}$ forms a subgroup of $F^{\times}$ of index d. In addition, for example, by Theorem (1.3) in Chapter III of [11], the set of norms from $K^{\times}$ into $F^{\times}$ forms a subgroup of $F^{\times}$ of index d. It follows that the set of norms of $K^{\times}$ in $F^{\times}$ is exactly the set of all elements of $F^{\times}$ of the form $c_0\pi^n$, where $v_F(c_0) = 0$, the projection of c_0 in $R/\mathfrak{p}$ is a dth power and $n \in \mathbf{Z}$. The lemma then follows immediately.

LEMMA 7.2. *Assume the notation of Lemma* 7.1. *Let* q_0 *be* p *raised to the odd part of the exponent of* p *in* q. *Then,* $|R/\mathfrak{p}|$, *the cardinality of the residue field, is a square if and only if* $\sigma_{q_0} \notin \mathrm{Galr}(\lambda)$.

Proof. The cardinality of a p-adic residue field of $\mathbf{Q}(\lambda)$ is simply p raised to the order of σ_p as an element of $\mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$. The cardinality of a p-adic residue field of $\mathrm{Galr}(\lambda)'$ is likewise p^m, where m is the smallest positive integer such that $\sigma_p^m \in \operatorname{Galr}(\lambda)$. Since the p-adic residue fields of $\operatorname{Galr}(\lambda)'$ and of $\mathbf{Q}(\psi_\lambda)$ are isomorphic, it follows that the cardinality of the residue field of F is a square if and only if σ_p projects into an element of even order in $\operatorname{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})/\operatorname{Galr}(\lambda)$. Let $\widehat{\sigma_p} \in \operatorname{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})/\operatorname{Galr}(\lambda)$ be the projection of σ_p, and generally denote by $\hat{\tau}$ the projection into the quotient group of any element $\tau \in \operatorname{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$. By Definition 2.3 and Definition 2.6, we have $\sigma_q \in \operatorname{Galg}(\lambda)$. Write $q = p^{2^\alpha \beta}$, where α is a non-negative integer and β is a positive odd integer. We then have $q_0 = p^\beta$. Furthermore, since $\operatorname{Galg}(\lambda) \subseteq \operatorname{Galr}(\lambda)$, the order of $\widehat{\sigma_p}$ is a divisor of $2^\alpha\beta$.

Suppose that $\sigma_{q_0} \in \operatorname{Galr}(\lambda)$. Then the order of $\widehat{\sigma_p}$ is a divisor of β and therefore odd. It then follows that the order of the residue field is not a square, and the lemma holds in this case. Hence, suppose that $\sigma_{q_0} \notin \operatorname{Galr}(\lambda)$. Then, the order of $\widehat{\sigma_p}$ is not a divisor of β, and since it divides $2^\alpha\beta$ it follows that it is even. Hence, the cardinality of the residue field is a square in this case. Hence, the lemma holds in all cases.

THEOREM 7.3. *Let $\lambda \in F_n$. Let ψ_λ be any of the irreducible characters of $\mathbf{SL}(n, q)$ parameterized by λ. Let q_0 be p raised to the odd part of the exponent of p in q. Then, the p-local Schur index $\mathrm{m}_p(\psi_\lambda)$ is always 1 or 2. Furthermore, it is 2 if and only if all of the following hold*:

(1) *p is odd, $2 \leq n_2 \leq (p-1)_2$, q is a square, and there exists some $\alpha \in \widehat{F_1}$ such that $\sigma_{q_0}\lambda = \alpha\lambda$.*

(2) *The restriction of χ_λ to $\mathbf{Z}(\mathbf{GL}(n, q)) \simeq F_1$ is a multiple of $\widehat{F_1}$. Let m be the order of ρ. Then, $m_2 = (q-1)_2$.*

Proof. We assume the notation of Lemma 7.1. Suppose, first, that λ satisfies Conditions (1) and (2). Then, by (1) $\sigma_{q_0} \in \operatorname{Galr}(\lambda)$. It follows from Lemma 7.2 that $|R/\mathfrak{p}|$ is not a square. Completing the field $\mathbf{Q}(\psi_\lambda)$ under a p-adic valuation, we obtain, from Theorem 5.9, that the element of $\operatorname{Br}(F)$ associated with ψ_λ is $[K/F, \tau, -1]$, where τ is a generator of $\operatorname{Gal}(K/F)$. Furthermore, $\mathrm{m}_p(\psi_\lambda)$ is the order of this element of the Brauer group. It follows immediately then that, in our case, $\mathrm{m}_p(\psi_\lambda)$ is either 1 or 2. In the notation of Lemma 7.1, we have $d = p - 1$. Since $|R/\mathfrak{p}|$ is not a square, it follows that $\overline{-1} \in |R/\mathfrak{p}|$ is not a dth power. Hence, by Lemma 7.1, -1 is not a norm from K. Hence, $\mathrm{m}_p(\psi_\lambda) \neq 1$, which implies $\mathrm{m}_\rho(\psi_\lambda) = 2$. Hence, the theorem holds in this case. It only remains to show that if λ does not satisfy (1)–(2) then $\mathrm{m}_p(\psi_\lambda) = 1$.

Suppose λ does not satisfy (1)–(2) and $\mathrm{m}_p(\psi_\lambda) \neq 1$. If $p = 2$, n is odd, or $n_2 > (p-1)_2$, then, by Theorem 5.9(1), we have $\mathrm{m}_p(\psi_\lambda) = 1$. Hence, p is odd and $2 \leq n_2 \leq (p-1)_2$. Assume that q is a square. Then, if (2) of our theorem does not hold, then Theorem 5.9 yields $\mathrm{m}_p(\psi_\lambda) = 1$, a contradiction. Hence, in this case (2) must hold, and, by our assumption, it follows that there does not exist any $\alpha \in \widehat{F_1}$ such that $\sigma_{q_0} \lambda = \alpha \lambda$. In other words, we have $\sigma_{q_0} \notin \mathrm{Galr}(\lambda)$. By Lemma 7.2, this implies that $|R/\mathfrak{p}|$ is a square. Hence, $\overline{-1} \in R/\mathfrak{p}$ is a $(p-1)$th power. Furthermore, by Theorem 5.9, we must have that -1 is not a norm from K, but this contradicts Lemma 7.1. This contradiction shows that q is not a square.

Suppose now that for any element $\beta \in \widehat{F_1}$ of order n_2 we have $\beta\lambda = \lambda$. Then, by Theorem 5.9, we must have again that -1 is not a norm from K. However, in the notation of Lemma 7.1, we have $d = (p-1)/2$, and it follows that $\overline{-1}$ is the dth power of some element of $R/\mathfrak{p}$. Hence, by Lemma 7.1, we have that -1 is a norm from K. This contradiction shows that, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$.

By Theorem 4.8, the field of values of ψ_λ is contained in $\mathbf{Q}(\lambda)$. Let L be a completion of $\mathbf{Q}(\lambda)$ under some extension of the valuation v_F, so that $F \subseteq L$. By Theorem 5.9(5) there exists some element $\gamma \in L^\times$ such that, for all $\sigma \in \mathrm{Gal}(L/F)$, we have

$$\sigma(\gamma) = \delta(\sigma)\gamma,$$

$\gamma^{p-1} \in F^\times$, and $[\psi_\lambda] = [K/F, \tau, \gamma^{p-1}]$ as an element of $\mathrm{Br}(F)$, where τ is a generator of $\mathrm{Gal}(K/F)$. Multiplying γ by any integral power of p does not affect its listed properties. Since p is unramified in the extension $L/\mathbf{Q}_p$, it follows that we may assume without loss that γ is such that $v_L(\gamma) = 0$, where v_L is the normalized valuation of L.

Let q_1 be the cardinality of the residue field of F. Then, by the argument of the proof of Lemma 7.2, it follows that $\sigma_{q_1} \in \mathrm{Galr}(\lambda)$ and that q_1 is the smallest power of p greater than 1 such that $\sigma_{q_1} \in \mathrm{Galr}(\lambda)$. By Definition 2.3 and Definition 2.6, we know that $\sigma_q \in \mathrm{Galg}(\lambda) \subseteq \mathrm{Galr}(\lambda)$. Hence, σ_q is a power of σ_{q_1}. Since q is not a square, we have that $\sigma_q = \sigma_{q_1}^u$, for some odd integer u. By Definition 5.7, $\delta(\sigma_{q_1})$ is a 2^αth root of 1, for some 2^α a divisor of $p-1$. Furthermore, since $\sigma_q \in \mathrm{Galg}(\lambda)$, we have $\delta(\sigma_q) = 1$. Since δ is a cross homomorphism (Lemma 5.8) and σ_{q_1} fixes $\delta(\sigma_{q_1})$, it follows that $\delta(\sigma_1) = (\delta(\sigma_{q_1}))^u = 1$. Since u is odd, this implies that $\delta(\sigma_{q_1}) = 1$. Hence, $\sigma_{q_1}(\gamma) = \gamma$. This implies that $\bar{\gamma}$ can be thought of as a non-zero element of the residue field of F. Hence, $\overline{\gamma^{p-1}}$ is the $(p-1)$th power of some element of the residue field of F. Hence, by Lemma 7.1, it follows that γ^{p-1} is a norm from K. Hence, $\mathrm{m}_p(\psi_\lambda) = 1$. This contradicts our assumption and, therefore, concludes the proof of the theorem.

COROLLARY 7.4. *Let $\lambda \in F_n$. Let ψ_λ be any of the irreducible characters of* $\mathbf{SL}(n, q)$ *parameterized by λ. Assume that ψ_λ can be thought of as being a character of* $\mathbf{PSL}(n, q)$. *Then*, $\mathrm{m}_p(\psi_\lambda) = 1$.

Proof. Suppose $\mathrm{m}_p(\psi_\lambda) \neq 1$. By Theorem 7.3, p is odd and n is even, and ρ has order divisible by $(q-1)_2$. By Lemma 5.10, it follows immediately that ψ_λ cannot be viewed as a character of $\mathbf{PSL}(n,q)$. Hence, the corollary holds.

8. r-LOCAL SCHUR INDICES

In this section we describe the local Schur indices $\mathrm{m}_r(\psi_\lambda)$ for each irreducible character $\psi_\lambda \in \mathbf{Irr}(\mathbf{SL}(n,q))$, where r is a (finite rational) prime different from p, the prime divisor of q.

DEFINITION 8.1. We let r be a (finite rational) prime with $r \neq p$, and we let $\lambda \in \mathscr{F}$. Suppose p is odd, $2 \leq n_2 \leq (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$. Then, we define $\mathbf{Q}(\lambda, r)$, $\mathrm{RGalr}(\lambda, r)$, and δ_r: $\mathrm{RGalr}(\lambda, r) \to \mathbf{C}^\times$ as follows:

(1) Suppose r is odd. Then, we set $\mathbf{Q}(\lambda, r) = \mathbf{Q}(\lambda)$, and we define $\mathrm{RGalr}(\lambda, r)$ to be the subgroup of $\mathrm{Galr}(\lambda)$ of those elements that fix every r'th root of unity. Furthermore, we define δ_r: $\mathrm{RGalr}(\lambda, r) \to \mathbf{C}^\times$ to be the restriction of δ (see Definition 5.7) to $\mathrm{RGalr}(\lambda, r)$.

(2) Suppose $r = 2$. The restriction of χ_λ to $\mathbf{Z}(\mathbf{GL}(n,q)) \simeq F_1$ is a multiple of $\rho \in \widehat{F_1}$. Recall from Definition 5.7 that δ is defined after some $t \in F_1$ of order $2(p-1)_2/n_2$ has been fixed. Now fix further some $t_0 \in F_1$ such that $t_0^n = t^2$ and some $\zeta \in \mathbf{C}^\times$ such that $\zeta^2 = \rho(t_0)$. Then, we set $\mathbf{Q}(\lambda, 2) = \mathbf{Q}(\lambda)(\zeta)$ to be $\mathbf{Q}(\lambda)$ extended by ζ. We define $\mathrm{RGalr}(\lambda, 2)$ to be the subgroup of $\mathrm{Gal}(\mathbf{Q}(\lambda, 2)/\mathbf{Q})$ of those elements which fix every $2'$th root of unity and which, when restricted to $\mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$, are elements of $\mathrm{Galr}(\lambda)$. Finally, for $\sigma \in \mathrm{RGalr}(\lambda, 2)$, we let $\sigma_0 \in \mathrm{Galr}(\lambda)$ be its restriction, and we set

$$\delta_2(\sigma) = \delta(\sigma_0)\sigma(\zeta^{-1})/\zeta^{-1}.$$

LEMMA 8.2. *Assume the hypotheses of Definition* 8.1. *Then,* δ_r *is a group homomorphism and its values are in the set* $\{1, -1\}$. *Furthermore, if* $r = 2$, ζ *is a primitive* $(2m_2)$*th root of unity in* $\mathbf{Q}(\lambda, 2)$.

Proof. By Lemma 5.3, there exists some $x \in \mathbf{GL}(n,q)$ such that $\mathbf{det}(x) = t$ and $\chi_\chi(x) \neq 0$. Let γ be the value of the central character of χ_λ on the conjugacy class sum C of x. Then, $\gamma \neq 0$ and, since for all the summands y of C we have $\mathbf{det}(y) = t$, it follows from Proposition 5.8 that, for all $\sigma \in \mathrm{Galr}(\lambda)$, we have

$$\sigma(\gamma) = \delta(\sigma)\gamma.$$

Furthermore, γ^2 is the value of the central character on C^2. Let Z be the subgroup of $\mathbf{Z}(\mathbf{GL}(n,q))$ of order $(p-1)_2$. Then, t^2 is the determinant of some element of Z, say $z_0 \in Z$, and $\mathbf{det}(z_0) = t^2$. Let $\nu = \rho(z_0)$. If $r = 2$, we choose z_0 to be identified with t_0 so that we have $\zeta^2 = \nu$. In any case, ν is some m_2th root of unity. Let y and y' be any $\mathbf{GL}(n,q)$ conjugates of x. Then, $\mathbf{det}(yy') = t^2$. It follows that $yy' = z_0 s$, for some $s \in \mathbf{SL}(n,q)$. Furthermore, $\chi_\lambda(yy') = \nu\chi_\lambda(s)$. Hence, the value of the central character of χ_λ on C^2 is νf, where $f \in \mathbf{Q}(\mathrm{Res}^{\mathbf{GL}(n,q)}_{\mathbf{SL}(n,q)}(\chi_\lambda)) = \mathbf{Q}(\psi_\lambda)$; see Theorem 4.8. In other words, we have $\gamma^2 = \nu f$.

Suppose first that r is odd. Let $\sigma \in \mathrm{RGalr}(\lambda, r)$. Since ν is an m_2th root of unity, by Definition 8.1, we have $\sigma(\gamma^2) = \gamma^2$. It follows that $\delta(\sigma) \in \{1, -1\}$. Furthermore, since δ is a cross homomorphism (Proposition 5.8), it follows that the restriction of δ to $\mathrm{RGalr}(\lambda, r)$ is a group homomorphism. Hence, the theorem holds in this case.

Assume now that $r = 2$. Let $\sigma \in \mathrm{RGalr}(\lambda, 2)$. Let $\sigma_0 \in \mathrm{Galr}(\lambda)$ be the restriction of σ. Since $\gamma \in \mathbf{Q}(\lambda)$, it follows that

$$\delta(\sigma_0) = \sigma_0(\gamma)/\gamma = \sigma(\gamma)/\gamma.$$

Hence, for each $\sigma \in \mathrm{RGalr}(\lambda, 2)$,

$$\delta_2(\sigma) = \sigma(\gamma\zeta^{-1})/(\gamma\zeta^{-1}).$$

It is straightforward to check, from the above equation, that δ_2 is a crossed homomorphism. Furthermore, since $\gamma^2 = \nu f$ and $\zeta^2 = \nu$, it follows that $(\gamma\zeta^{-1})^2 = f \in \mathbf{Q}(\psi_\lambda)$. This implies that $\delta_2(\sigma) \in \{1, -1\}$, for all $\sigma \in \mathrm{RGalr}(\lambda, 2)$. Hence, δ_2 is a group homomorphism also in this case. This completes the proof of the lemma.

If $\mathfrak{p}$ is an even prime of the ground field, there can be quadratic extensions which are ramified at $\mathfrak{p}$ but whose discriminant has an even valuation by $\mathfrak{p}$. We could say, for lack of a better word, that, up to squares, the extension has an *odd discriminant*. (*Note*. Strictly speaking, the discriminant of such an extension is still divisible by $\mathfrak{p}$, but, up to squares, it is not.) For this reason, when $r = 2$, we will need to distinguish between two types of extensions.

DEFINITION 8.3. We let $\lambda \in \mathscr{F}$. Suppose p is odd, $2 \le n_2 \le (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$. Then we say that $\delta_2\colon \mathrm{RGalr}(\lambda, r) \to \mathbf{C}^\times$ has an even or odd discriminant according to the following: Let ζ_0 be a 2-element of largest possible order in $\mathbf{Q}(\lambda, 2)$. We let M_1 and M_2 be the set of all elements of $\mathbf{Q}(\zeta_0)$ which are fixed under the action of respectively

$\operatorname{RGalr}(\lambda, 2)$ and $\ker(\delta_2)$. By Definition 8.1 and Lemma 8.2, M_2 is an extension field of M_1, and its degree $[M_2 : M_1] = |\delta_2(\operatorname{RGalr}(\lambda, 2))|$ is at most 2. If $[M_2 : M_1] = 1$, we say that δ_2 has odd discriminant. Suppose $[M_2 : M_1] = 2$. Let Δ^2 be the discriminant of the extension. Since the extension $M_1/\mathbf{Q}$ is totally ramified at 2, we let v be the normalized valuation of M_1 dividing 2. Then, $v(\Delta^2)$ is uniquely determined modulo 2. We say that δ_2 has odd discriminant if $v(\Delta^2) \equiv 0 \pmod 2$, and we say that δ_2 has even discriminant if $v(\Delta^2) \equiv 1 \pmod 2$.

LEMMA 8.4. *Assume the hypotheses of Definition* 8.1. *Let some element* $\gamma \in \mathbf{Q}(\lambda)^\times$ *be such that, for all* $\sigma \in \operatorname{Galr}(\lambda)$, *we have* $\sigma(\gamma) = \delta(\sigma)\gamma$ *and* $\gamma^{p-1} \in \mathbf{Q}(\psi_\lambda)^\times$. *Let v be a normalized r-adic valuation on* $\mathbf{Q}(\psi_\lambda)$. *Then, the following hold*:

(1) *Suppose r is odd. Then*,

$$v(\gamma^{p-1}) \equiv (p-1)/|\delta_r(\operatorname{RGalr}(\lambda, r))| \pmod{p-1}.$$

(2) *Suppose $r = 2$. If δ_2 has an odd discriminant, then $p - 1$ divides* $v(\gamma^{p-1})$. *If δ_2 has an even discriminant then* $v(\gamma^{p-1}) \equiv (p-1)/2 \pmod{p-1}$.

Proof. Let η be an element of largest r'th order in $\mathbf{Q}(\lambda)$. We set $F = \mathbf{Q}(\psi_\lambda)(\eta)$ and $L = F(\gamma)$. Let v_F and let v_L be normalized valuations of F and L respectively, where v_L divides v_F, which in turn divides v. We set $K = \mathbf{Q}(\lambda, r)$. It follows from Theorem 4.8 and Definition 8.1 that $\operatorname{RGalr}(\lambda, r)$ is the Galois group of the extension K/F. The extension $F/\mathbf{Q}(\psi_\lambda)$ is unramified at v, since η has r'th order. Hence, we have

$$v(f) = v_F(f) \qquad \text{for all } f \in \mathbf{Q}(\psi_\lambda)^\times.$$

The extension L/F is totally ramified at v_F.

Consider first the case where r is odd. Then, it follows from Definition 8.1 that $\delta_r(\operatorname{RGalr}(\lambda, r))$ is isomorphic to $\operatorname{Gal}(L/F)$. Suppose $|\delta_r(\operatorname{RGalr}(\lambda, r))| = 1$. Then $\gamma \in F^\times$, and it follows that $v_F(\gamma^{p-1}) = (p-1)v_F(\gamma)$. Hence, $v(\gamma^{p-1}) = v_F(\gamma^{p-1})$ is divisible by $p - 1$. Therefore, the lemma holds in this case. Henceforth, we assume that $|\delta_r(\operatorname{RGalr}(\lambda, r))| \neq 1$. Observe that, by Lemma 8.2, we have $|\delta_r(\operatorname{RGalr}(\lambda, r))| = 2$, so that $[L : F] = 2$ and $\gamma^2 \in F^\times$. Now since L/F is totally ramified and v_F does not divide 2, it follows that $v_L(\gamma)$ is odd. Hence,

$$v(\gamma^{p-1}) = v_F(\gamma^{p-1}) = (p-1)v_L(\gamma)/2 \equiv (p-1)/2 \pmod{p-1}.$$

Hence, the lemma holds if r is odd.

Consider now the case where $r = 2$. We assume the notation of Definition 8.1. Let $\nu = \gamma\zeta^{-1}$, and let $E = F(\nu)$. It is straightforward to show that, for all $\sigma \in \mathrm{RGalr}(\lambda, 2)$, $\sigma(\nu) = \delta_2(\sigma)\nu$. Since δ_2 is a group homomorphism with values in the set $\{1, -1\}$ (Lemma 8.2), it follows that $\mathrm{Gal}(E/F)$ is isomorphic to $\delta_2(\mathrm{RGalr}(\lambda, 2))$ and that $\nu^2 \in F^\times$. Note that, since ζ is a root of unity,

$$v(\gamma^{p-1}) = v\left((\gamma\zeta^{-1})^{p-1}\right) = v(\nu^{p-1}).$$

Hence, if δ_2 is trivial then $E = F$, and it follows that

$$v(\nu^{p-1}) = v_F(\nu^{p-1}) = (p-1)v_F(\nu)$$

is a multiple of $p - 1$. Since by definition 8.2 has an odd discriminant in this case, the lemma holds if δ_2 is trivial. We assume henceforth that δ_2 is not trivial. By Lemma 8.2, δ_2 is a group homomorphism and its kernel has index 2. We have that E/F is a quadratic extension and that $\nu^2 \in F^\times$. Let M_1 and M_2 be as in Definition 8.3, and let $\Delta^2 \in M_1^\times$ be the discriminant of the extension M_2/M_1. We have $\mathbf{Q}(\zeta_0) \cap F = M_1$ and $\mathbf{Q}(\zeta_0) \cap E = M_2$. Hence, Δ^2 is equal to ν^2 up to squares in $F^\times$. In particular, we have $v_F(\Delta^2) \equiv v_F(\nu^2) \pmod 2$. It follows that

$$\begin{aligned} v(\gamma^{p-1}) &= v_F(\nu^{p-1}) = (p-1)v_F(\nu^2)/2 \\ &\equiv (p-1)v_F(\Delta^2)/2 \pmod{p-1}. \end{aligned}$$

Since 2 is fully ramified in $M_1/\mathbf{Q}$ and v_F is an even valuation, v_F divides the unique even valuation v_M of M_1. Since $F = M_1(\eta)$, v_M is unramified in the extension F/M_1. Therefore, $v_F(\Delta^2) = v_M(\Delta^2)$. The lemma then follows immediately from Definition 8.3.

DEFINITION 8.5. We let r be a (finite rational) prime with $r \neq p$, and we let $\lambda \in \mathscr{F}$.

(1) We denote by $\mathrm{Ram}(\lambda, r)$ the subgroup of $\mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$ of all those elements that fix every r'th root of unity.

(2) We denote by σ_r the element of $\mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$ which fixes every r-element of $\mathbf{Q}(\lambda)^\times$ and which raises every r'th root of unity in $\mathbf{Q}(\lambda, r)$ to its rth power.

LEMMA 8.6. *Let $\lambda \in \mathscr{F}_n$. Let r be a (finite rational) prime with $r \neq p$. Let F be an r-adic completion of the field of values $\mathbf{Q}(\psi_\lambda)$ of $\psi_\lambda \in$ $\mathbf{Irr}(\mathbf{SL}(n,q))$. We let v_F be the discrete valuation of F normalized so that $v_F(F^\times) = \mathbf{Z}$. Let η_p be a primitive pth root of 1 and let $K = F(\eta_p)$. Let $d = [K : F]$ and let $a \in F^\times$. Then a is a norm from K if and only if d divides $v_F(a)$.*

Proof. The extension K/F is Galois and unramified, and its degree is d. We let v_K be the normalized valuation on K. Let $b \in K^{\times}$, and let c be the norm of b. Then $v_K(c) = dv_K(b)$. As K/F is unramified, it follows that $v_F(c) = v_K(c)$ is a multiple of d. Hence, the norm of b is an element of $F^{\times}$ whose valuation is divisible by d. The set of all such elements of $F^{\times}$ forms a subgroup of $F^{\times}$ of index d. In addition, for example, by Theorem (1.3) in Chapter III of [11], the set of norms from $K^{\times}$ into $F^{\times}$ forms a subgroup of $F^{\times}$ of index d. It follows that the set of norms of $K^{\times}$ in $F^{\times}$ is exactly the set of all elements of $F^{\times}$ whose valuation is divisible by d. The lemma then follows immediately.

LEMMA 8.7. *Assume the notation of Lemma* 8.6. *Suppose p is odd, $2 \leq n_2 \leq (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$. Then, $d_2 = (p-1)_2$ if and only if both of the following hold*:

(1) *r is a non-square modulo p.*

(2) *Let $\overline{\sigma_r}$ be the image of σ_r in* $\mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})/\mathrm{Ram}(\lambda, r)\,\mathrm{Galr}(\lambda)$. *Then, the order of $\overline{\sigma_r}$ is odd.*

Proof. By Theorem 4.8, we have that $\mathbf{Q}(\psi_\lambda) = \mathrm{Galr}(\lambda)'$. Let r^{α} be the order of the residue field of F. Then α is the order of $\overline{\sigma_r}$. Furthermore, since K/F is unramified, the order of the residue field of K is $r^{d\alpha}$. Let β be the multiplicative order of r modulo p. Then, since $K = F(\eta_p)$, $d\alpha$ is the least common multiple of β and α. It is a standard result that $\beta_2 = (p-1)_2$ if and only if r is a non-square modulo p. Hence, $d_2 = (p-1)_2$ if and only if (1) holds and α is odd. However, since α is the order of $\overline{\sigma_r}$, α is odd if and only if (2) holds. This concludes the proof of the lemma.

THEOREM 8.8. *Let r be some (finite rational) prime different from p. Let $\lambda \in F_n$. Let ψ_λ be any of the irreducible characters of* $\mathbf{SL}(n, q)$ *parameterized by λ. Then the r-local Schur index* $\mathrm{m}_r(\psi_\lambda)$ *is always* 1 *or* 2. *Furthermore, it is* 2 *if and only if all of the following hold*:

(1) *p is odd, $2 \leq n_2 \leq (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$.*

(2) *r is a non-square modulo p.*

(3) *Let $\overline{\sigma_r}$ be the image of σ_r in* $\mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})/\mathrm{Ram}(\lambda, r)\,\mathrm{Galr}(\lambda)$ (*see Definition* 8.5). *Then, the order of $\overline{\sigma_r}$ is odd.*

(4) *The map δ_r:* $\mathrm{RGalr}(\lambda, r) \to \{1, -1\}$ (*see Definition* 8.1 *and Lemma* 8.2) *is not trivial.*

(5) *If $r = 2$, then δ_2 has an even discriminant* (*see Definition* 8.3).

Proof. Assume the notation of Lemma 8.6. Suppose first that (1) is not satisfied. Then, by Theorem 5.9, either $\mathrm{m}_r(\psi_\lambda) = 1$ or the element of the Brauer group $\mathrm{Br}(F)$ associated with ψ_λ is $[K/F, \tau, -1]$, where τ is a generator of the Galois group of K/F. However, $v_F(-1) = 0$, so that it follows from Lemma 8.6 that -1 is a norm from K. Hence, in any case, $\mathrm{m}_r(\psi_\lambda) = 1$. It follows that the theorem holds in the case where (1) is not satisfied. We assume henceforth that (1) holds.

We assume the notation of Lemma 8.7. By Theorem 5.9, there exists some $\gamma \in \mathbf{Q}(\lambda)^\times$ such that $\gamma^{p-1} \in F^\times$ and $[\psi_\lambda] = [K/F, \tau, \gamma^{p-1}] \in \mathrm{Br}(F)$. By Lemma 8.4, $v_F(\gamma^{p-1})$ is divisible by $(p-1)/2$. Suppose that either (2) or (3) does not hold. Then, by Lemma 8.7, $d_2 \neq (p-1)_2$. Hence, d divides $(p-1)/2$, and, by Lemma 8.6, γ^{p-1} is a norm from K. This implies that $\mathrm{m}_r(\psi_\lambda) = 1$. Hence, the theorem holds if either (2) or (3) does not hold. Henceforth, we assume that (2) and (3) hold. By Lemma 8.7, it follows that $d_2 = (p-1)_2$.

Suppose (4) does not hold. Then δ_r is trivial, and it follows from Definition 8.3 and Lemma 8.4 that $v_F(\gamma^{p-1})$ is divisible by $p-1$. Since d divides $p-1$, it follows from Lemma 8.6 that γ^{p-1} is a norm from K and $\mathrm{m}_r(\psi_\lambda) = 1$. Hence, the theorem also holds in this case. We assume henceforth that (4) holds. Analogously, if (5) does not hold, then $r = 2$, and by Lemma 8.4 followed by Lemma 8.6 we again obtain $\mathrm{m}_r(\psi_\lambda) = 1$. Hence, the theorem also holds if (5) does not hold. We assume henceforth that (5) holds.

In the case we have left, namely, when (1)–(5) hold, Lemma 8.4 yields that $v_F(\gamma^{p-1}) \equiv (p-1)/2 \pmod{p-1}$. Since $d_2 = (p-1)_2$, it follows that d does not divide $v_F(\gamma^{p-1})$. By Lemma 8.6, this tells us that γ^{p-1} is not a norm from K. Hence, $\mathrm{m}_r(\psi_\lambda) \neq 1$. Since d divides $p-1$ and $p-1$ divides $v_F((\gamma^{p-1})^2)$, then d divides $v_F((\gamma^{p-1})^2)$. It follows that $(\gamma^{p-1})^2$ is a norm from K and $\mathrm{m}_r(\psi_\lambda)$ divides 2. Hence, $\mathrm{m}_r(\psi_\lambda) = 2$. This completes the proof of the theorem.

9. GLOBAL PROPERTIES

Theorem 5.9, above, provides for each irreducible character $\psi \in \mathbf{Irr}(\mathbf{SL}(n, q))$ a representative for the class $[\psi] \in \mathrm{Br}(\mathbf{Q}(\psi))$ in the Brauer group of $\mathbf{Q}(\psi)$. We have seen that $\mathrm{m}_p(\psi) \leq 2$, for p any rational prime or ∞ (a fact originally proved by Gow [5]). Hence, $[\psi]$ can also be characterized by the values of $\mathrm{m}_p(\psi)$ for all p a rational prime or ∞ (see, for example, [3, Theorem 2.14]). In this section, we describe for each ψ the set of all p such that $\mathrm{m}_p(\psi) \neq 1$ and obtain some properties of these sets.

DEFINITION 9.1. Let G be a finite group and let $\psi \in \mathbf{Irr}(G)$. We set

$$M(\psi) = \{p : p \text{ is a rational prime or } \infty \text{ and } \mathrm{m}_p(\psi) \neq 1\}.$$

LEMMA 9.2. *Let $\lambda \in \mathscr{F}$. Suppose that δ_2 is defined for λ and δ_2 has an even discriminant* (*see Definition* 8.1 *and Definition* 8.3). *Let $\nu \in \mathbf{Q}(\lambda)^{\times}$ be any* 2-*element. Then there exists some $\sigma \in \operatorname{Galr}(\lambda)$ such that σ fixes every element of odd order in $\mathbf{Q}(\lambda)^{\times}$ and $\delta(\sigma) \neq \sigma(\nu)/\nu$* (*see Definition* 5.7). *In particular, the restriction of δ to the subgroup of all the elements of* Galr(λ) *which fix every element of odd order in $\mathbf{Q}(\lambda)^{\times}$ is not identically* 1.

Proof. Suppose that, for all $\sigma_0 \in \operatorname{Galr}(\lambda)$ such that σ_0 fixes every element of odd order in $\mathbf{Q}(\lambda)^{\times}$, we have $\delta(\sigma_0) = \sigma_0(\nu)/\nu$, where $\nu \in \mathbf{Q}(\lambda)^{\times}$ is a fixed 2-element. Let ζ be as in Definition 8.1. Then ζ is a 2-element in $\mathbf{Q}(\lambda, 2)^{\times}$, and, for every $\sigma \in \operatorname{RGalr}(\lambda, 2)$, letting $\sigma_0 \in \operatorname{Galr}(\lambda)$ be its restriction, we have $\delta_2(\sigma) = \delta(\sigma_0)\sigma(\zeta^{-1})/\zeta^{-1}$. It then follows from our hypothesis that, setting $\mu = \nu\zeta^{-1}$, we have $\delta_2(\sigma) = \sigma(\mu)/\mu$, for all $\sigma \in \operatorname{RGalr}(\lambda, 2)$. Here, $\mu = \nu\zeta^{-1} \in \mathbf{Q}(\lambda, 2)^{\times}$ is a fixed 2-element. Setting M_1 and M_2 as in Definition 8.3, we see that $\mu^2 \in M_1$ and $M_2 = M_1(\mu)$. Since δ_2 has an even discriminant, Definition 8.3 implies that μ^2 has an odd 2-adic valuation, against the fact that it is a root of unity. This contradiction completes the proof of the lemma.

Next, we describe for every irreducible character $\psi \in \mathbf{Irr}(\mathbf{SL}(n, q))$ an easily computable finite set M such that $M(\psi) \subseteq M$. This, together with the earlier results giving $\mathrm{m}_p(\psi)$ for all p, provides both a description of $[\psi]$ as an element of $\operatorname{Br}(\mathbf{Q}(\psi))$ and the rational Schur index $\mathrm{m}_{\mathbf{Q}}(\psi)$. In order not to complicate the statement of the theorem, we do not try use the smallest possible set M. In particular, Theorem 5.9 provides a number of instances where $M(\psi)$ is immediately known to be the empty set, but we do not incorporate the full strength of these results into our next theorem.

THEOREM 9.3. *Let $\lambda \in F_n$. Let ψ_λ be any of the irreducible characters of $\mathbf{SL}(n, q)$ parameterized by λ. Then, the following hold*:

(1) *If $p = 2$, or n is odd, or $n_2 > (p-1)_2$, then $M(\psi_\lambda) = \varnothing$.*

(2) *Suppose p is odd, $2 \leq n_2 \leq (p-1)_2$, and q is a square. Then, $M(\psi_\lambda) \subseteq \{\infty, p\}$.*

(3) *Suppose p is odd, $2 \leq n_2 \leq (p-1)_2$, q is not a square, and, for some* (*hence for all*) *element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$. Then, $M(\psi_\lambda) \subseteq \{\infty\}$.*

(4) *Suppose p is odd, $2 \leq n_2 \leq (p-1)_2$, q is not a square, and, for some* (*hence for all*) *element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$. Let $P(\lambda)$ be the set of all* (*finite*) *rational primes r, such that r is non-square modulo p*

and there exist $\sigma \in \mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$ *and* $\alpha \in \widehat{F_1}$ *such that* σ *has order a power of* 2 *and fixes every element of* r' *order in* $\mathbf{Q}(\lambda)^\times$, $\sigma\lambda = \alpha\lambda$, *and the order of* α *is divisible by* n_2. *Then* $M(\psi_\lambda) \subseteq P(\lambda) \cup \{\infty\}$.

Hence, for each λ, *we have an easily computable finite set* M *such that* $M(\psi_\lambda) \subseteq M$. *For each element* $r \in M$, *we then can compute* $m_r(\psi_\lambda)$ *using Theorem* 6.5, *Theorem* 7.3, *and Theorem* 8.8. *This then computes* $M(\psi_\lambda)$ *as the set of all* $r \in M$ *such that* $m_r(\psi_\lambda) \neq 1$. *Hence, this describes explicitly* $[\psi_\lambda]$ *as an element of* $\mathrm{Br}(\mathbf{Q}(\psi))$ *in terms of its invariants. Finally, if* $M(\psi_\lambda) = \varnothing$ *then* $m_{\mathbf{Q}}(\psi_\lambda) = 1$. *Otherwise,* $m_{\mathbf{Q}}(\psi_\lambda) = 2$.

Proof. (1) follows immediately from Theorem 5.9(1). Suppose p is odd, $2 \le n_2 \le (p-1)_2$, and q is a square. Then, by Theorem 8.8, we have that $\mathrm{m}_r(\psi_\lambda) = 1$ for all finite rational primes r different from p. Hence, (2) holds. Suppose p is odd, $2 \le n_2 \le (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda = \lambda$. Then, by Theorem 7.3 and Theorem 8.8, if r is any finite rational prime, then $\mathrm{m}_r(\psi_\lambda) = 1$. Hence (3) holds.

Finally, suppose p is odd, $2 \le n_2 \le (p-1)_2$, q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$. Let r be a finite rational prime such that $\mathrm{m}_r(\psi_\lambda) \neq 1$. We need to show that $r \in P(\lambda)$. By Theorem 7.3, $\mathrm{m}_p(\psi_\lambda) = 1$, so that $r \neq p$. By Theorem 8.8(2, 4, and 5), this implies that r is a non-square modulo p; δ_r is not trivial; and that, if $r = 2$, δ_2 has an even discriminant. If r is odd, by Definition 8.1, there exists some $\sigma \in \mathrm{Galr}(\lambda)$ that fixes every element of r' order in $\mathbf{Q}(\lambda)^\times$ and such that $\delta(\sigma) \neq 1$. By Lemma 9.2, such an element σ exists also when $r = 2$. If $r = 2$, then σ has order a power of 2. If $r \neq 2$, then, by Lemma 8.2, $\delta(\sigma) = \delta_r(\sigma) = -1$ and δ_r is a group homomorphism, which implies that, by replacing σ by an odd power of itself if necessary, we may assume that σ has order a power of 2. Hence, the order of σ is a power of 2 in any case. By Definition 3.1, $\sigma \in \mathrm{Galr}(\lambda)$ just means that there exists some $\alpha \in \widehat{F_1}$ such that $\sigma\lambda = \alpha\lambda$. By Definition 5.7, $\delta(\sigma) \neq 1$ then implies that the order of α is divisible by n_2. This shows that $r \in P(\lambda)$ and completes the proof of the theorem.

Walter Feit has studied some global properties of the centralizer algebra for each irreducible character of some families of finite groups. In [3, 4] Feit proves that, for every character ψ of each of the families he considers, he has $|M(\psi)| \le 2$. These families include all finite simple groups of order less than 10^6, the covering groups of each of the sporadic simple groups, and other quasi-simple groups. In contrast, $|M(\psi)|$ is unbounded when ψ runs through the characters of the double covers of the alternating groups [15]. The work of Gow [6] left open the question of whether or not $|M(\psi)|$

could be bigger than 2 for $\psi \in \mathbf{Irr}(\mathbf{SL}(n, q))$. We now study $M(\psi)$, for $\psi \in \mathbf{Irr}(\mathbf{SL}(n, q))$. We prove, in particular, that $|M(\psi)|$ is unbounded.

THEOREM 9.4. *Let p be any odd prime and let q be any odd power of p. Let $M = \{r_1, \dots, r_s\}$ be a set of odd primes different from p. Assume that r_i is a non-square modulo p, for $i = 1, \dots, s$. Assume, in addition, that the number of primes r_i for which p is a non-square modulo r_i is even. Then there exists some n and some $\lambda \in \mathscr{F}_n$ such that $\psi_\lambda \in \mathbf{Irr}(\mathbf{SL}(n, q))$ has rational values and $M \subseteq M(\psi_\lambda) \subseteq M \cup \{2, \infty\}$. Furthermore, if r_i does not divide $(q-1)$ for $i = 1, \dots, s$ and $s \geq 3$, then we may view ψ_λ as an irreducible character of $\mathbf{PSL}(n, q)$.*

Proof. We only need to consider the case $s \geq 1$. Let $\mathbf{sgn} \in \widehat{F_1}$ be the element of order 2. Let d be the smallest positive integer such that $r_1 \cdots r_s$ divides $q^d - 1$, and let $\rho\colon F_d \to \mathbf{C}^\times$ be a linear character such that $|\rho(F_d)| = r_1 \cdots r_s$. By Definition 2.1, $\rho \in \mathscr{G}_d$. For $i = 1, \dots, s$, let η_i be a primitive r_ith root of unity. Then, $\mathbf{Q}(\rho) = \mathbf{Q}(\eta_1, \dots, \eta_s)$. We set

$$G_i = \mathrm{Gal}\big(\mathbf{Q}(\eta_1, \dots, \eta_s)/\mathbf{Q}(\eta_1, \dots, \eta_{i-1}, \eta_{i+1}, \dots, \eta_s)\big).$$

Then, Gal $(\mathbf{Q}(\rho)/\mathbf{Q})$ is the direct product $G_1 \times \cdots \times G_s$. Each G_i is cyclic of even order. We define δ to be the unique linear character of $\mathrm{Gal}(\mathbf{Q}(\rho)/\mathbf{Q})$ which restricts to each G_i as its linear character of order 2.

We now define $\lambda \in \mathscr{F}$ as follows. If $\sigma \in \ker(\delta)$, we let $\lambda(\sigma\rho) = (1)$ be the unique partition of 1. If $\sigma \in \mathrm{Gal}(\mathbf{Q}(\rho)/\mathbf{Q})$ but $\sigma \notin \ker(\delta)$, then we set $\lambda(\mathbf{sgn}\ \sigma\rho) = (1)$ (see Definition 3.1 for the meaning of the product of $\mathbf{sgn}$ with an element of $\mathscr{G}_d$). If $s > 1$ or $r_1 \equiv 1 \pmod 4$, we further set $\lambda(1) = \lambda(\mathbf{sgn}) = (1)$. If $s = 1$ and $r_1 \equiv -1 \pmod 4$, we set, instead, $\lambda(1) = \lambda(\mathbf{sgn})$ to be the empty partition. In any case, we assign the empty partition to all the other elements of $\mathscr{G}$.

It follows from our definition of λ that, for each $\sigma \in \mathrm{Gal}(\mathbf{Q}(\rho)/\mathbf{Q})$, we have $\sigma\lambda = \lambda$ if $\sigma \in \ker(\delta)$, and $\sigma\lambda = \mathbf{sgn}\ \lambda$ if $\sigma \notin \ker(\delta)$. Using the direct product decomposition of $\mathrm{Gal}(\mathbf{Q}(\rho)/\mathbf{Q})$, we may write σ_q as a product $\tau_1 \cdots \tau_s$, where $\tau_i \in G_i$. The 2-part of the order of τ_i is $(r_i - 1)_2$ if and only if q is a non-square modulo r_i if and only if p is a non-square modulo r_i. Hence, $\delta(\tau_i) = -1$ if and only if p is a non-square modulo r_i. Since the number of such r_i is even by hypothesis, it follows that $\delta(\sigma_q) = 1$, which implies $\sigma_q \lambda = \lambda$. Hence, $\lambda \in \mathscr{F}$ (Definition 2.3).

Let $n = \mathbf{deg}(\lambda)$ (Definition 2.3). If $s > 1$ or $r_1 \equiv 1 \pmod 4$, then $n = 2 + (r_1 - 1) \cdots (r_s - 1)$. If $s = 1$ and $r_1 \equiv -1 \pmod 4$, then $n = (r_1 - 1)$. In any case, $n_2 = 2$. It follows from the definition of λ that $\mathbf{sgn}\ \lambda \neq \lambda$. Hence, the hypotheses of Definition 5.7 are satisfied. We pick $t \in F_1$ of order $2(p-1)_2/2 = (p-1)_2$. From Definition 3.1 we have $\mathrm{Galr}(\lambda) =$

$\mathrm{Gal}(\mathbf{Q}(\rho)/\mathbf{Q})$. Furthermore, it follows from our choice of t that the δ of Definition 5.7 coincides with the δ we constructed above.

By Theorem 4.8, we have $\mathbf{Q}(\psi_\lambda) = \mathbf{Q}$. Since q is not a square, it follows from Theorem 7.3, that $\mathrm{m}_p(\psi_\lambda) = 1$. By Definition 8.1, $\mathrm{RGalr}(\lambda, r_i) = G_i$ and δ_{r_i} has order 2, for $i = 1, \dots, s$. Theorem 8.8 then immediately implies that for odd finite primes r we have $\mathrm{m}_r(\psi_\lambda) = 2$ if and only if $r \in M$.

Finally, suppose that r_i does not divide $(q - 1)$ for $i = 1, \dots, s$ and $s \geq 3$. Let $\mu = \mathrm{Res}_{F_1}^{F_d}(\mathbf{sgn}\ \sigma\rho)$, for some $\sigma \in \mathrm{Galr}(\mathbf{Q}(\lambda)/\mathbf{Q})$, but $\sigma \notin \ker(\delta)$. Since the order of μ divides both $q - 1$ and the order of $\mathbf{sgn}\ \rho$, the order of μ is at most 2. It follows that μ is independent of our choice of σ. For similar divisibility reasons, $\mathrm{Res}_{F_1}^{F_d}(\sigma\rho) = 1$, for all $\sigma \in \mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$. The support of λ consists of $\mathbf{sgn}\ \sigma\rho$ for $\sigma \in \mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q})$, but $\sigma \notin \ker(\delta)$, and $\sigma\rho$, for $\sigma \in \ker(\delta)$. The action of σ_q stabilizes both of these sets. Let α be the number of orbits of σ_q in its action on the set

$$S = \{\mathbf{sgn}\ \sigma\rho : \sigma \in \mathrm{Gal}(\mathbf{Q}(\lambda)/\mathbf{Q}) \text{ but } \sigma \notin \ker(\delta)\}.$$

By Proposition 2.5, the restriction of χ_λ to $\mathrm{Z}(\mathbf{GL}(n, q))$ is μ^α. The number of elements of S is $|S| = (r_1 - 1) \cdots (r_s - 1)/2$. The orbits of σ_q on S all have the same number of elements in them, namely, the multiplicative order β of q modulo $r_1 \cdots r_s$. Since this order is the lowest common multiple of the multiplicative order of q modulo each of the r_i, it follows that the 2-part of β divides $r_i - 1$ for some i, say, $\beta_2 \mid r_1 - 1$. Since $s \geq 3$, it follows that $\alpha = |S|/\beta$ is even. Hence, $\mathrm{Z}(\mathbf{GL}(n, q))$ is in the kernel of χ_λ and ψ_λ can be viewed as a character of $\mathbf{PSL}(n, q)$ (see Lemma 5.10). This concludes the proof of the theorem.

We now introduce some notation for the statement of the next corollary. For G any finite group, we set

$$MM(G) = \max\{|M(\psi)| : \psi \in \mathbf{Irr}(G)\}$$

to be the maximum cardinality of all the sets $M(\psi)$ as ψ runs through the irreducible characters of G.

COROLLARY 9.5. *Let p be any odd prime and let q be any power of p. Then if q is a square, then $MM(\mathbf{SL}(n, q)) \leq 2$. If q is odd and q is not a square, then*

$$\limsup_{n \to \infty} (MM(\mathbf{SL}(n, q))) = \limsup_{n \to \infty} (MM(\mathbf{PSL}(n, q))) = \infty.$$

Proof. Suppose q is a square. Then, by Theorem 9.3, $MM(\mathbf{SL}(n, q)) \leq 2$, and the lemma holds in this case. Suppose q is not a square. Using Dirichlet's Theorem, we may obtain, for arbitrarily large s, distinct odd primes $r_1, \dots, r_s$, different from p, and such that r_i is a non-square modulo p and not a divisor of $q-1$, for $i = 1, \dots, s$. Using Dirichlet's theorem and quadratic reciprocity, we may also find a larger prime r' which is a non-square modulo p and is not a divisor of $q-1$, and for which p modulo r' is also a non-square. Adding the new prime r' if necessary to $r_1, \dots, r_s$, we may assume, in addition, that the number of primes r_i for which p is a non-square modulo r_i is even. Set $M = \{r_1, \dots, r_s\}$. Then, by Theorem 9.4, there exists some n and some $\lambda \in \mathscr{F}_n$ such that $\psi_\lambda \in \mathbf{Irr}(\mathbf{PSL}(n, q)) \subseteq \mathbf{Irr}(\mathbf{SL}(n, q))$ has rational values and $M \subseteq M(\psi_\lambda)$. This shows that there exist n with $MM(\mathbf{PSL}(n, q))$ arbitrarily large. This concludes the proof of the corollary.

These results show that $|M(\psi)|$ can be arbitrarily large for $\psi \in \mathbf{Irr}(\mathbf{SL}(n, q))$, if we let n vary. Our next goal is to show that, on the other hand, $|M(\psi)|$ is always bounded above by n.

LEMMA 9.6. *Suppose $\lambda \in \mathscr{F}$ and $\sigma \in \mathrm{Galr}(\lambda)$ has order a power of 2. Then, there exists some $\alpha \in \widehat{F_1}$, such that $\sigma\lambda = \alpha\lambda$ and, for each odd prime r dividing the order of α, there exists some r-element $a \in \mathbf{Q}(\lambda)^\times$ such that $\sigma(a) \neq a$.*

Proof. Assume the lemma is false, and let λ and σ provide a counterexample. Let P be the set whose elements are 2 and all the primes r for which there exists some r-element $a \in \mathbf{Q}(\lambda)^\times$ such that $\sigma(a) \neq a$. By Definition 3.1, there exists some $\alpha \in \widehat{F_1}$ such that $\sigma\lambda = \alpha\lambda$. Among all such α, we choose one whose order is divisible by the smallest number of primes not in P. Let Q be the set of prime divisors of the order of α which are not in P. By the minimality of our counterexample the cardinality of Q is as small as possible, but not zero. Pick some $s \in Q$.

Write $\alpha = \beta\gamma$, where $\beta, \gamma \in \widehat{F_1}$ are respectively the s-part and the s'-part of α. Since $s \notin P$, σ fixes β and σ normalizes each Hall subgroup of $\widehat{F_1}$. Using repeatedly the fact that $\sigma\lambda = \alpha\lambda$, we obtain that $\lambda = \beta^o\gamma'\lambda$, where o is the order of σ (a power of 2) and γ' is such that any prime t dividing its order satisfies $t \neq s$ and $t \in P \cup Q$. Since the order of β is odd, this implies that $\beta\lambda = \gamma''\lambda$, where γ'' is again such that any prime t dividing its order satisfies $t \neq s$ and $t \in P \cup Q$. Then, we have $\alpha\lambda = \gamma''\gamma\lambda$, where $\gamma''\gamma$ is such that any prime t dividing its order satisfies $t \neq s$ and $t \in P \cup Q$. This contradicts our choice of α and completes the proof of the lemma.

PROPOSITION 9.7. *Let q be any power of a prime, n a positive integer, and $\psi \in \mathbf{Irr}(\mathbf{SL}(n, q))$. Then, the number of finite primes in $M(\psi)$ is a most $n-1$. In particular, $|M(\psi)| \leq n$.*

Proof. Assume that the proposition is false, and let ψ be a counterexample. Let $\lambda \in \mathcal{F}_n$ be such that ψ is $\mathbf{GL}(n, q)$ conjugate to ψ_λ. If ψ is a linear character, then $M(\psi) = \varnothing$, and $|M(\psi)| \leq n - 1$, a contradiction. Hence, ψ is non-linear and, in particular, $n \geq 2$. Hence, by Theorem 9.3, we have that p is odd, $2 \leq n_2 \leq (p-1)_2$, and q is not a square, and, for some (hence for all) element $\beta \in \widehat{F_1}$ of order n_2, we have $\beta\lambda \neq \lambda$.

Let $\beta \in \widehat{F_1}$ be of order n_2. Then we have $\beta\lambda \neq \lambda$. Hence, there exists some $\theta \in \mathcal{G}$ such that $\lambda(\theta)$ is not the empty partition and $\lambda(\theta) \neq \lambda(\beta^{-1}\theta)$. Let m be such that $\theta \in \widehat{F_m}$. We let $t \in F_1$ be as in Definition 5.7 and we let $t_0 \in F_m$ be a 2-element whose norm is t. Furthermore, we set $\nu = \theta(t_0)$. Hence, ν is a 2-element of $\mathbf{Q}(\lambda)^\times$.

Suppose $2 \in M(\psi)$. Then, by Theorem 8.8, δ_2 has an even discriminant. Hence, we may apply Lemma 9.2, which yields that there exists some $\tau_2 \in \mathrm{Galr}(\lambda)$ such that τ_2 fixes every element of odd order in $\mathbf{Q}(\lambda)^\times$ and $\delta(\tau_2) \neq \tau_2(\nu)/\nu$. Since the order of τ_2 is a power of 2, by Lemma 9.6, there exists some $\alpha_2 \in \widehat{F_1}$, a 2-element, such that $\tau_2\lambda = \alpha_2\lambda$. This implies that $\alpha_2^{-1}\tau_2\lambda(\theta) = \lambda(\theta)$, so that $\tau_2^{-1}(\alpha_2\theta)$ is in the support of λ. Set $\theta_2 = \tau_2^{-1}(\alpha_2\theta) \in \widehat{F_m}$. If $\theta_2 = \theta$, then $\tau_2^{-1}(\alpha_2(t)\theta(t_0)) = \theta(t_0)$, which implies $\delta(\tau_2)\nu = \tau_2(\nu)$, a contradiction. Hence, $\theta_2 \neq \theta$. In addition, the odd part of θ_2 coincides with the odd part of θ.

Suppose $r \in M(\psi)$ and r is a finite odd prime. By Theorem 8.8, there exists some $\tau_r \in \mathrm{RGalr}(\lambda, r)$, a 2-element, such that $\delta(\tau_r) = -1$. By Lemma 9.6 there exists $\alpha_r \in \widehat{F_1}$, a $\{2, r\}$-element, such that $\tau_r\lambda = \alpha_r\lambda$. By Definition 5.7, $\alpha_r(t) = -1$, which implies that n_2 divides the order of α_r. Set $\theta_r = \tau_r^{-1}(\alpha_r\theta) \in \widehat{F_m}$ and $\phi_r = \theta_r\theta^{-1}$. Since $\tau_r \in \mathrm{RGalr}(\lambda, r)$, r is odd, and n_2 divides the order of α_r, we also have that n_2 divides the order of ϕ_r. Since $\alpha_r^{-1}\tau_r\lambda(\theta) = \lambda(\theta)$, we have that $\lambda(\theta_r) = \lambda(\theta)$, so that θ_r is in the support of λ. Furthermore, then $\{2, r\}'$-part of θ_r coincides with the $\{2, r\}'$-part of θ.

Suppose, for some finite odd $r \in M(\psi)$, the r-part of θ_r equals the r-part of θ. Since the order of ϕ_r is divisible by n_2, a power of it, ϕ_r^a, say, can be identified with β^{-1} when viewed as an element of $\widehat{F_m}$. Repeated applications of the identity $\alpha_r^{-1}\tau_r\lambda = \lambda$ then yield, in particular, that $\lambda((\tau_r^{-1}\alpha_r)^a\theta) = \lambda(\theta)$. But $(\tau_r^{-1}\alpha_r)^a\theta = (\tau_r^{-1}\alpha_r)^{a-1}(\phi_r\theta) = \phi_r^a\theta$, since the order of ϕ_r is not divisible by r. Since $\phi_r^a = \beta^{-1}$, this contradicts our choice of θ. Hence, for each finite odd $r \in M(\psi)$, the r-part of θ_r is different from the r-part of θ.

Now the elements θ_r, for $r \in M(\psi)$ a finite odd prime, are all distinct and different from θ, as well as different from θ_2 if $2 \in M(\psi)$. Hence, the set $\{\theta\} \cup \{\theta_r : r \text{ is finite, } r \in M(\psi)\}$ is contained in the support of λ and has cardinality at least one more than the number of finite elements of

$M(\psi)$. This implies that $n = \mathbf{deg}(\lambda)$ is at least one more than the number of finite elements of $M(\psi)$. The proposition then follows.

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